# THE CHROMATIC STRUCTURE OF DENSE GRAPHS

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# About this Thesis

The research in this thesis was carried out in the Department of Pure Mathematics and Mathematical Statistics at the University of Cambridge between October 2018 and July 2021, and was supervised by Professor Andrew Thomason. Chapter 7 is based on joint work with Ewan Davies.

I confirm that this thesis is the result of my own work and if that work was the outcome of collaboration, then it is declared both above and in the text. It is not substantially the same as any work that has already been submitted for any degree or other qualification.

Freddie Illingworth

# Synopsis

This thesis focusses on extremal graph theory, the study of how local constraints on a graph affect its macroscopic structure. We primarily consider the chromatic structure: whether a graph has or is close to having some (low) chromatic number.

Chapter 2 is the slight exception. We consider an induced version of the classical Turán problem. Introduced by Loh, Tait, Timmons, and Zhou, the induced Turán number $\mathrm{ex}(n, \{H, F\text{-ind}\})$ is the greatest number of edges in an $n$-vertex graph with no copy of $H$ and no induced copy of $F$. We asymptotically determine $\mathrm{ex}(n, \{H, F\text{-ind}\})$ for $H$ not bipartite and $F$ neither an independent set nor a complete bipartite graph. We also improve the upper bound for $\mathrm{ex}(n, \{H, K_{2,t}\text{-ind}\})$ as well as the lower bound for the clique number of graphs that have some fixed edge density and no induced $K_{2,t}$.

The next three chapters form the heart of the thesis. Chapters 3 and 4 consider the Erdős-Simonovits question for locally $r$-partite graphs: what are the structure and chromatic number of graphs with large minimum degree and where every neighbourhood is $r$-colourable? Chapter 3 deals with the locally bipartite case and Chapter 4 with the general case.

While the subject of Chapters 3 and 4 is a natural local to global colouring question, it is also essential for determining the minimum degree stability of $H$-free graphs, the focus of Chapter 5. Given a graph $H$ of chromatic number $r+1$, this asks for the minimum degree that guarantees that an $H$-free graph is close to $r$-partite. This is analogous to the classical edge stability of Erdős and Simonovits. We also consider the question for the family of graphs to which $H$ is not homomorphic, showing that it has the same answer.

Chapter 6 considers sparse analogues of the results of Chapters 3 to 5 obtaining the thresholds at which the sparse problem degenerates away from the dense one.

Finally, Chapter 7 considers a chromatic Ramsey problem first posed by Erdős: what is the greatest chromatic number of a triangle-free graph on $n$ vertices or with $m$ edges? We improve the best bounds known and obtain tight (up to a constant factor) bounds for the list chromatic number, answering a question of Cames van Batenburg, de Joannis de Verclos, Kang, and Pirot.

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# CONTENTS

# Chapter 1

# Introduction and Preliminaries

## 1.1 Dense Problems for $H$-free Graphs and Other Families

Much of this thesis examines the structure and stability of monotone families of graphs and, in particular, considers Turán and Erdős-Simonovits-type problems. This section introduces these and ties in the relevant chapters. By a *monotone* family of graphs we mean one that is closed under deleting edges and vertices (that is, under taking subgraphs): such a family is characterised by the minimal graphs it does not contain. The canonical monotone family is that of *$H$-free* graphs (for some fixed graph $H$): all those graphs that do not contain $H$ as a subgraph.

### 1.1.1 Turán Problems

Some of the first milestones in extremal graph theory were the resolution of various Turán-type problems. At their most general, these ask for the maximum number of a substructure in an $n$-vertex graph (or other combinatorial structure) if it does not contain certain forbidden substructures. The original and most basic question asked for $\mathrm{ex}(n, H)$ – the most edges in an $n$-vertex $H$-free graph – as well as for the $H$-free graphs that attain this maximum. As early as 1907, Mantel [Man07] showed that if a graph with $n$ vertices is triangle-free, then it has at most $\lfloor n^2/4 \rfloor$ edges. Here, the only extremal graph is the complete bipartite graph with parts as equal in size as possible, $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.

Turán [Tur41] solved the problem for cliques: the unique $n$-vertex $K_{r+1}$-free graph with most edges is the complete $r$-partite graph with parts as equal in size as possible. This graph is commonly called the *Turán graph* and is denoted $T_r(n)$. Its size can be conveniently bounded:

$$\left(1-\tfrac{1}{r}\right)\binom{n}{2} < e(T_r(n)) \leqslant \left(1-\tfrac{1}{r}\right)\tfrac{n^2}{2},$$

and so the Turán graph has $(1-1/r+o(1))\binom{n}{2}$ edges. When $H$ is not a clique, life is more complicated. The extremal graphs are often not Turán and in many cases are not even known. For non-bipartite $H$, Erdős and Simonovits [ES66] determined the answers to leading order. Making use of a theorem of Erdős and Stone [ES46], they showed that it is the chromatic number, $\chi(H)$, of $H$ which asymptotically determines $\mathrm{ex}(n, H)$. Further, in a series of papers [Erd67a; Erd68; Sim68] they independently proved that all graphs which are close to extremal are close to Turán. The following is a simplified précis of some of their results.

**Theorem 1.1** (Erdős & Simonovits). *Let $H$ be a graph with chromatic number $r+1$. Then*

$$\mathrm{ex}(n, H) = \left(1-\tfrac{1}{r}+o(1)\right)\binom{n}{2}.$$

*Furthermore, any $n$-vertex $H$-free graph with $(1-1/r+o(1))\binom{n}{2}$ edges can be obtained from the Turán graph $T_r(n)$ by deleting and adding at most $o(n^2)$ edges.*

Much less is established for bipartite $H$. For the most part, asymptotically tight bounds are elusive and Theorem 1.1 is of little use: it only gives a $o(n^2)$ upper bound, while in fact the classical theorem of Kővári, Sós, and Turán [KST54] gives a much better bound of $\mathrm{ex}(n, K_{s,t}) = \mathcal{O}_{s,t}(n^{2-1/s})$. For more on the degenerate Turán problem, see the survey of Furëdi and Simonovits [FS13].

The original Turán problem has been generalised in many directions. These include to other families of graphs, to non-complete host graphs, and to hypergraphs. For hypergraphs, there is a cornucopia of unanswered problems – see Keevash's survey [Kee11].

One direction, taken by Loh, Tait, Timmons, and Zhou [LTTZ18], was to forbid an induced subgraph. Asking for the greatest number of edges in an $n$-vertex graph with no induced $H$ does not give an interesting answer: provided $H$ is not a clique, then the answer is $\binom{n}{2}$ as witnessed by the complete graph $K_n$. Instead, they also forbid a (non-induced) subgraph which stops the complete graph being extremal. They defined the *induced Turán number*

$$\mathrm{ex}(n, \{H, F\text{-ind}\})$$

to be the maximum number of edges in an $n$-vertex graph with no copy of $H$ and no induced copy of $F$. We address the problem of determining induced Turán numbers in Chapter 2. First we asymptotically determine them for general $H$ and $F$ in § 2.2.

**Theorem 2.1.** *Let $H$ and $F$ be graphs with chromatic numbers $r+1$ and $k+1$, respectively. Then*

$$\mathrm{ex}(n, \{H, F\text{-ind}\}) = \begin{cases} \left(1 - \frac{1}{r} + o(1)\right)\binom{n}{2} & \text{if } k \geqslant r \text{ or } F \text{ is not complete multipartite,} \\ \left(1 - \frac{1}{k} + o(1)\right)\binom{n}{2} & \text{if } k < r \text{ and } F \text{ is complete multipartite.} \end{cases}$$

A case of interest that remains is complete bipartite $F$ (where the previous theorem just gives $\mathrm{ex}(n, \{H, F\text{-ind}\}) = o(n^2)$). Particular attention [LTTZ18; NTT18; EGM19] has previously been paid to $F = K_{2,t}$. In § 2.6 we improve the general upper bound for $\mathrm{ex}(n, \{H, K_{2,t}\text{-ind}\})$, stating only the special case of complete $H$ here.

**Theorem 2.6.** *For any integers $t \geqslant 2$ and $r \geqslant 1$,*

$$\mathrm{ex}(n, \{K_{r+1}, K_{2,t}\text{-ind}\}) < \frac{t}{2\sqrt{t-1}} R(t,r)^{1/2} n^{3/2},$$
$$\mathrm{ex}(n, \{K_{r+1}, K_{2,t}\text{-ind}\}) \leqslant \frac{1}{2}\big(R(t,r) - 1 + o(1)\big)^{1/2} n^{3/2}.$$

For the definition of the Ramsey number $R(t,r)$, see § 1.2.1. Our results are flexible enough to allow $H$ to grow with $n$. In § 2.4 we obtain lower bounds for the clique number of $n$-vertex graphs with no induced $K_{2,t}$ and $\alpha\binom{n}{2}$ edges. These have the correct growth rate in $n$. This generalises results of Gyárfás, Hubenko, and Solymosi [GHS02] and Holmsen [Hol20] for the clique number of $n$-vertex graphs with no induced $C_4$ and $\alpha\binom{n}{2}$ edges.

### 1.1.2 Erdős-Simonovits Problems

For a fixed graph $H$ with chromatic number $r+1 \geqslant 3$, Theorem 1.1 gives the structure (to leading order) of $H$-free graphs with $(1 - 1/r + o(1))\binom{n}{2}$ edges. It is natural to wonder what happens below: are $H$-free graphs with fewer than $(1 - 1/r + o(1))\binom{n}{2}$ edges close to $r$-partite? The answer is no in a strong sense: for any $c < 1 - 1/r$ and positive integer $k$, there are $n$-vertex $H$-free graphs with at least $c\binom{n}{2}$ edges but which require the deletion of $\Omega_c(n^2)$ edges to be made $k$-partite. We construct such a graph as follows. First, by considering a connected component of $H$ with chromatic number $r+1$, we may assume that $H$ is connected. Since $H$ is not bipartite, it contains an odd cycle – denote its length by $\ell$. One of the first applications of the probabilistic method was Erdős's [Erd59] construction of graphs with arbitrarily high girth and chromatic number. In particular, there is an $H$-free graph $H'$ with chromatic number greater than $k$ and girth greater than $\ell$. Take $n$ very large and fix $\alpha \in (0,1)$ (chosen later). Let $G$ be the disjoint union of the Turán graph $T_r(\alpha n)$ and the balanced blow-up (defined in § 1.3) of $H'$ on $(1-\alpha)n$ vertices. The balanced blow-up of $H'$ contains no odd cycles of length $\ell$ and the Turán graph is $r$-partite, so $G$ is $H$-free. Furthermore, making $G$ $k$-partite requires the removal of all copies of $H'$ from $G$ – a simple counting argument, given in Lemma 1.7, shows that $\Omega_\alpha(n^2)$ edges need deleting. Finally, $G$ has at least

$$e(T_r(\alpha n)) \geqslant \left(1 - \tfrac{1}{r}\right)\tbinom{\alpha n}{2}$$

edges. By taking $\alpha$ sufficiently close to one (in terms of $c$ but not $n$), one may ensure this last quantity is at least $c\binom{n}{2}$.

In this construction the high chromatic number has been squirrelled away into a bad graph (the blow-up of $H'$) on a small proportion of the vertices. To avoid this squirrelling away, we might insist that our graph has large minimum degree, say linear in the number of vertices. In 1973, Erdős and Simonovits [ES73] did exactly this when they posed the following question. For a fixed graph $H$ and positive integer $k$, what $\delta$ guarantees that every $n$-vertex $H$-free graph with minimum degree greater than $\delta n$ is $k$-colourable? The values of $\delta$ as $k$ varies form the *chromatic profile of H-free graphs*.

An Erdős-Simonovits problem considers a general family of graphs $\mathcal{F}$ and wants upper bounds on the chromatic number of its members that have large minimum degree. To be concrete, it asks for the *chromatic profile of* $\mathcal{F}$: the sequence of values $\delta_\chi(\mathcal{F}, k)$ where

$$\delta_\chi(\mathcal{F}, k) = \inf\{d \colon \text{if } \delta(G) \geqslant d|G| \text{ and } G \in \mathcal{F}, \text{ then } \chi(G) \leqslant k\}.$$

Any $n$-vertex graph with minimum degree greater than $cn$ has more than $c\binom{n}{2}$ edges. Hence the high minimum degree constraint of the Erdős-Simonovits question is stronger than the corresponding many edge one of the Turán problem. However, it does lead to suitably stronger and richer conclusions. Andrásfai, Erdős, and Sós [AES74] proved the following seminal and quintessential result (see Brandt [Bra03] for a particularly short proof).

**Theorem 1.2** (Andrásfai-Erdős-Sós)**.** *Let $r \geqslant 2$ and $G$ be a $K_{r+1}$-free graph with minimum degree greater than*

$$\left(1 - \frac{1}{r - 1/3}\right)|G|.$$

*Then $G$ is $r$-colourable. Furthermore, $1 - 1/(r - 1/3)$ is tight.*

This says that $\delta_\chi(\mathcal{F}, r) = 1 - 1/(r - 1/3)$ where $\mathcal{F}$ is the family of $K_{r+1}$-free graphs. Note that the $K_{r+1}$-free graph with greatest minimum degree (and most edges), the Turán graph $T_r(n)$, has minimum degree $(1 - 1/r)n$. In particular, the theorem gives information about graphs whose minimum degrees are $\Omega(n)$ away from that of the extremal graph – we are far from the regime of Theorem 1.1.

The chromatic profile of triangle-free graphs has been extensively studied (see § 3.1 for more details). This asks for bounds on the chromatic number of triangle-free graphs with large minimum degree and is a very natural local to global colouring question: every neighbourhood is 1-colourable and neighbourhoods are large, so it seems likely the whole graph will have small chromatic number. Indeed this is true for minimum degree down to $1/3 \cdot n$. In Chapters 3 and 4, we study a generalisation of this problem: what is the chromatic profile of graphs in which every neighbourhood is $r$-colourable? Such graphs are called locally $r$-partite.

**Definition 1.3.** *Let $r$ be a positive integer. A graph is **locally $r$-partite** if the neighbourhood of every vertex induces an $r$-colourable graph.*

Chapter 3 focusses on locally bipartite graphs. The Turán graph $T_3(n)$ is locally bipartite, while any $n$-vertex graph with more edges contains a $K_4$ by Turán's theorem [Tur41] and so is not locally bipartite. Thus, all locally bipartite graphs have minimum degree at most $2/3 \cdot n$ and those with that minimum degree are 3-colourable. We give structural and colouring results down to minimum degree $8/15 \cdot n$ (below $1/2 \cdot n$ locally bipartite graphs can have arbitrarily large chromatic number as shown by Łuczak and Thomassé [ŁT10]). The following theorem gives a sense of the sort of results we obtain. We are using $\overline{C}_7$ to denote the complement of the 7-cycle – note this is locally bipartite. Balanced blow-ups and homomorphisms are defined in § 1.3.

**Theorem 1.4.** *Let $G$ be a locally bipartite graph.*

- *If $\delta(G) > 4/7 \cdot |G|$, then $G$ is 3-colourable. Balanced blow-ups of $\overline{C}_7$ show that this is tight.*
- *If $\delta(G) > 5/9 \cdot |G|$, then $G$ is homomorphic to $\overline{C}_7$. Also, $G$ is either 3-colourable or contains $\overline{C}_7$.*
- *If $\delta(G) > 6/11 \cdot |G|$, then $G$ is 4-colourable.*

Chapter 4 considers locally $b$-partite graphs for $b \geqslant 3$. Again, by Turán's theorem, these all have minimum degree at most $(1 - 1/(b+1))n$ and those with that minimum degree are $(b+1)$-colourable. We show that this extends down at least as far as $1 - 1/(b + 1/7)$. Furthermore, we show that the threshold at which locally $b$-partite graphs go from having bounded to unbounded chromatic number is $1 - 1/b$.

**Theorem 1.5.** *Let $b \geqslant 3$ be an integer and $G$ be a locally $b$-partite graph.*

- *If $\delta(G) > (1 - 1/(b + 1/7)) \cdot |G|$, then $G$ is $(b+1)$-colourable.*
- *For any $\varepsilon > 0$, there is an absolute constant $C$ such that if $\delta(G) > (1 - 1/b + \varepsilon) \cdot |G|$, then $G$ is $C$-colourable.*

- *There are locally $b$-partite graphs $G$ with minimum degree $(1 - 1/b - o(1)) \cdot |G|$ and arbitrarily large chromatic number.*

The chromatic profile of locally colourable graphs, while being interesting in its own right, also is, perhaps surprisingly, essential for determining the minimum degree stability of $H$-free graphs. At the start of this subsection we saw that edge stability of $H$-free graphs occurs at $(1 - 1/r + o(1))\binom{n}{2}$ but not below. We will see that things are different for minimum degree stability. The most basic question is, given an $(r+1)$-chromatic graph $H$, to determine

$$\delta_H = \inf\{c\colon \text{if } |G| = n,\ \delta(G) \geqslant cn, \text{ and } G \text{ is } H\text{-free}, \\ \text{then } G \text{ can be made } r\text{-partite by deleting } o(n^2) \text{ edges}\},$$

that is, to determine the minimum degree that guarantees the same conclusion as edge stability. This can also be viewed as an approximate Erdős-Simonovits problem for $H$-free graphs: rather than asking for $r$-colourable, close to will suffice. Chapter 5 studies this, placing it within an approximate chromatic profile, as well as considering the question for the family of graphs to which $H$ is not homomorphic, showing that has the same answer, $\delta_H$.

As an example, consider $H$ being a triangle, $K_3$. By Turán's theorem, any triangle-free graph has minimum degree at most $1/2 \cdot n$ and those with that minimum degree are bipartite. Dropping the minimum degree slightly, triangle-free graphs remain bipartite until one gets to minimum degree $2/5 \cdot n$ (this is exactly the $r = 2$ case of the Andrásfai-Erdős-Sós theorem). At this point, 5-cycles appear and a 'phase transition' occurs: a balanced $n$-vertex blow-up of a 5-cycle has minimum degree $2/5 \cdot n$, is triangle-free, and is far from bipartite by Lemma 1.7. In conclusion, $\delta_{K_3}$ is $2/5$.

In contrast to edge stability, $\delta_H$ does not just depend on the chromatic number of $H$. For $H = C_5$, a balanced blow-up of a 5-cycle is not $H$-free, so does not show $\delta_H \geqslant 2/5$. In fact, the phase transition for $C_5$-free graphs does not occur until $2/7$ at which point balanced blow-ups of 7-cycles appear. More generally, the following theorem from § 5.1 determines $\delta_H$ for 3-chromatic $H$.

**Theorem 5.2** ($\delta_H$ for 3-chromatic $H$)**.** *Let $H$ be a 3-chromatic graph. There is a smallest positive integer $g$ such that $H$ is not homomorphic to $C_{2g+1}$. Then*

$$\delta_H = \frac{2}{2g+1}.$$

This says that, for 3-chromatic $H$, $\delta_H$ is determined by the first odd cycle to which $H$ is not homomorphic. Is there such a sequence of graphs for higher chromatic numbers? For each $r \geqslant 3$, we find a sequence of eleven graphs such that $\delta_H$, for $(r+1)$-chromatic $H$, is determined by the first graph in the sequence to which $H$ is not homomorphic (see Theorem 5.3). For those $H$ homomorphic to all eleven, we sadly do not determine $\delta_H$, but do bound it between $1-1/(r-1)$ and $1-1/(r-6/7)$. Such an $H$ would be rare (indeed, every $H$ in which one is commonly interested is not homomorphic to all eleven).

## 1.2 Sparser Problems

Non-bipartite Turán and Erdős-Simonovits problems are dense problems: they concern graphs with $\Omega(n^2)$ edges. Our final two chapters study sparser problems.

Chapter 6 considers sparse analogues of the problems in Chapters 3 to 5. Many classical extremal problems ask questions about (spanning) subgraphs of the $n$-vertex complete graph, $K_n$, such as how many edges can such a subgraph have if it is triangle-free. In the last two decades, there has been extensive study of these same extremal problems hosted not on $K_n$ but on the binomial random graph $G(n,p)$. This is a subgraph of $K_n$ where every edge is present independently with probability $p$. These new problems are termed *sparse* or *random* analogues of the originals.

The sparse analogues often exhibit threshold behaviour: for $p$ above the threshold, the sparse problem behaves like an approximate version of the dense one and, for $p$ below, the problem degenerates away. As an example, consider the question of the most edges in a triangle-free subgraph of $G(n,p)$. Of course, Mantel's [Man07] answer to the corresponding dense problem is that the greatest number of edges in a triangle-free subgraph of $K_n$ is $\lfloor n^2/4 \rfloor = (1/2+o(1))\binom{n}{2}$. Frankl and Rödl [FR86] showed that, for $p \geqslant n^{-1/2+o(1)}$, the greatest number of edges in a triangle-free subgraph of $G(n,p)$ is $(1/2+o(1))p\binom{n}{2}$ – that is, the problem behaves like the dense one (note that $p\binom{n}{2}$ is the expected number of edges in $G(n,p)$). On the other hand, if $p = o(n^{-1/2})$, then the expected number of triangles in $G(n,p)$ is at most $(pn)^3 = o(pn^2)$. Hence, by removing one edge from each triangle, one obtains a triangle-free subgraph of $G(n,p)$ with $(1-o(1))p\binom{n}{2}$ edges. Degeneracy has occurred and being triangle-free is no longer a meaningful constraint when $p = o(n^{-1/2})$.

The sparse analogue of $\delta_H$, for an $(r+1)$-chromatic graph $H$, is

$$\delta_{H,p} = \inf\{c\colon \text{if } G \text{ is an } H\text{-free spanning subgraph of } G(n,p) \text{ with } \delta(G) \geqslant cpn,$$
$$\text{then } G \text{ can be made } r\text{-partite by deleting } o(pn^2) \text{ edges}\},$$

where $pn$ and $pn^2$ appear as these are the expected growth rate of the degrees and number of edges in $G(n,p)$, respectively. In § 6.2, we show amongst other things that

$$\delta_{H,p} = \begin{cases} \delta_H & \text{if } p = \omega(n^{-1/m_2(H)}), \\ 1 & \text{if } \omega(\log n/n) = p = o(n^{-1/m_2(H)}), \end{cases}$$

where $m_2(H)$ is the *2-density* of $H$ (see Definition 6.2). For $p = o(\log n/n)$ the situation degenerates even further: $G(n,p)$ will almost surely have some isolated vertices, so every spanning subgraph will have minimum degree zero.

We see that above a threshold this sparse analogue behaves like the dense version and degeneracy occurs below: it is possible to make the graph $H$-free by deleting $o(pn)$ edges incident to each vertex. The result for $\delta_{H,p}$ follows from a reasonably routine application of standard techniques. However, our results on locally colourable graphs require more care as infinitely many subgraphs are forbidden (for example, locally bipartite graphs are exactly those containing no odd wheel).

We show that the sparse problem for locally bipartite graphs exhibits threshold behaviour around $n^{-1/2}$. For $p \geqslant n^{-1/2+o(1)}$, the problem behaves like the dense one and our results from Chapter 3 carry over. Here is an example of the sort of results possible.

**Theorem 1.6.** *Let $p \geqslant n^{-1/2+o(1)}$ and $G$ be a locally bipartite spanning subgraph of $G(n,p)$. The following hold asymptotically almost surely.*

- *If $\delta(G) \geqslant (4/7 + o(1))pn$, then $G$ can be made 3-colourable by deleting $o(pn^2)$ edges. Furthermore, $4/7$ is tight.*
- *If $\delta(G) \geqslant (5/9 + o(1))pn$, then $G$ can be made homomorphic to $\overline{C}_7$ by deleting $o(pn^2)$ edges.*
- *If $\delta(G) \geqslant (6/11 + o(1))pn$, then $G$ can be made 4-colourable by deleting $o(pn^2)$ edges.*

On the other hand, for $p \leqslant (2n)^{-1/2}$, degeneracy occurs (see Theorem 6.15).

The waters are murkier for locally $b$-partite graphs ($b \geqslant 3$). There is not a nice characterisation of these in terms of forbidden subgraphs and it is not clear what threshold to speculate. We do know that, for sufficiently large $p$, the sparse problem is not degenerate.

**Theorem 6.6.** *Let $b \geqslant 3$ be an integer and let $b' = (b+15)/2 - 90/(b+12)$. If $p = \omega(n^{-1/b'})$ and $G$ is a locally $b$-partite spanning subgraph of $G(n,p)$ with $\delta(G) \geqslant (1 - 1/(b + 1/7))pn$, then $G$ can be made $(b+1)$-colourable by deleting $o(pn^2)$ edges.*

For sufficiently small $p$ (say $p \geqslant (2n)^{-1/2}$), degeneracy must occur simply because it does so for locally bipartite graphs, which are a subfamily of locally $b$-partite ones.

### 1.2.1 $\chi$-Ramsey Problem for Triangle-free Graphs

Ramsey theory is often referred to as finding order within disorder. More concretely, a Ramsey problem normally involves colouring (the edges of) some combinatorial structure and asking for some monochromatic (or other colour configuration) substructure.

The classic Ramsey problem asks, given graphs $G$ and $H$, what is the smallest $n$ such that if the edges of $K_n$ are red-blue coloured, then there will either be a red copy of $G$ or a blue copy of $H$. That such an $n$ exists follows from a theorem of Ramsey [Ram30] and the least such $n$ is denoted by $R(G, H)$.

Special consideration has been given to $G$ and $H$ being cliques: for $G = K_s$ and $H = K_t$, we write $R(s, t)$ for $R(G, H)$. Viewing red edges as present and blue ones as not, we see that $R(s, t)$ is the least $n$ such that any $n$-vertex graph has either an $s$-clique or an independent set of size $t$. Erdős and Szekeres [ES35] proved the seminal upper bound

$$R(s,t) \leqslant \binom{s+t-2}{t-1}.$$

For many years the best lower bounds were only polynomial in $s$ and $t$. The first exponential lower bound for $R(s, s)$ was proven by Erdős [Erd47] as one of the first applications of his highly influential probabilistic method.

These upper and lower bounds give $\sqrt{2}^{(1+o(1))s} \leqslant R(s,s) \leqslant 4^{(1-o(1))s}$. Despite significant efforts, this state of affairs remains today with only improvements in the $o(1)$ terms to report: by Spencer [Spe75] for the lower bound and by Rödl (unpublished), Thomason [Tho88], Conlon [Con09], and Sah [Sah20] for the upper bound.

There has been greater success for off-diagonal Ramsey numbers where $s$ is fixed and $t$ is allowed to grow. The $s = 3$ case has been a particular triumph as the asymptotic value of $R(3, t)$ is now known to within a factor of four. $R(3, t)$ is the least $n$ for which any $n$-vertex triangle-free graph must contain an independent $t$-set. Determining this is equivalent to finding the smallest independence number amongst $n$-vertex triangle-free graphs.

The colour classes of vertex-colourings are independent sets, so the chromatic number, $\chi(G)$, and independence number, $\alpha(G)$, of a graph $G$ satisfy

$$\alpha(G) \cdot \chi(G) \geqslant |G|.$$

In light of this, there is a natural ‘chromatic Ramsey’ question that Erdős [Erd67b] first asked in 1967: what is the greatest chromatic number amongst $n$-vertex triangle-free graphs? He raised the problem of determining the following two functions

$$f(n) = \max\{\chi(G) \colon G \text{ is triangle-free}, |G| = n\},$$
$$g(m) = \max\{\chi(G) \colon G \text{ is triangle-free}, e(G) = m\},$$

and showed that

$$\begin{aligned} f(n) &= \Omega(n^{1/2}/\log n), \\ f(n) &= \mathcal{O}(n^{1/2}), \\ g(m) &= \Omega(m^{1/3}/\log m), \\ g(m) &= \mathcal{O}(m^{1/3}). \end{aligned}$$

In Chapter 7, we improve the best upper bounds known for $f(n)$ and $g(m)$ as well as for the list chromatic versions, confirming a conjecture of Cames van Batenburg, de Joannis de Verclos, Kang, and Pirot [CdKP20].

The best bounds known for $f(n)$ are

$$(1/\sqrt{2} - o(1))\sqrt{n/\log n} \leqslant f(n) \leqslant (2\sqrt{2} + o(1))\sqrt{n/\log n}.$$

The lower bound is a corollary of a recent groundbreaking result proved independently by Bohman and Keevash [BK21] and Fiz Pontiveros, Griffiths, and Morris [FGM20]. They followed the triangle-free process to its asymptotic end showing that the resulting $n$-vertex triangle-free graph has independence number at most $(\sqrt{2} + o(1))\sqrt{n \log n}$. Another corollary of this amazing result is that $R(3,t) \geqslant (1/4 - o(1))t^2/\log t$. The upper bound follows by combining an observation of Erdős and Hajnal [EH85] and Shearer's [She83] bound $R(3,t) \leqslant (1 + o(1))t^2/\log t$. Erdős and Hajnal observed that iteratively pulling out the large independent sets guaranteed by Shearer's result and giving each one a different colour will colour any $n$-vertex triangle-free graph with $(2\sqrt{2} + o(1))\sqrt{n/\log n}$ colours.

There is a factor of four between the upper and lower bounds for $f(n)$. The first result of Chapter 7 is an improvement to the upper bound by a factor of $\sqrt{2}$. Rather than just removing large independent sets, we remove large neighbourhoods until the maximum degree is not too big and apply a recent colouring result of Molloy (Theorem 7.4).

**Theorem 7.1.** *Let $G$ be a triangle-free graph on $n$ vertices. Then*

$$\chi(G) \leqslant (2 + o(1))\sqrt{n/\log n}.$$

We also give a corresponding improvement to the upper bound for $g(m)$ (see Theorem 7.3).

For the list colouring version (what is the greatest list chromatic number amongst $n$-vertex triangle-free graphs?), the best lower bound known is again given by the triangle-free process: $(1/\sqrt{2} - o(1))\sqrt{n/\log n}$. However, Erdős and Hajnal's method no longer applies and the best upper bound known is $\mathcal{O}(\sqrt{n})$ due to Cames van Batenburg et al [CdKP20]. We confirm a conjecture of theirs by showing that the correct growth rate is $\Theta(\sqrt{n/\log n})$.

**Theorem 7.2.** *Let $G$ be a triangle-free graph on $n$ vertices. Then*

$$\chi_\ell(G) \leqslant (4\sqrt{2} + o(1))\sqrt{n/\log n}.$$

We do likewise for the edge problem (see Theorem 7.3).

## 1.3 Notation, Blow-ups, and Homomorphisms

A simple yet surprisingly useful notion is that of a graph blow-up. Given a graph $G$, a *blow-up of* $G$ is a graph obtained by replacing each vertex $v$ of $G$ by a **non-empty** independent set $I_v$ and each edge $uv$ by a complete bipartite graph between classes $I_u$ and $I_v$. We say that a vertex $v$ *has been blown-up by* $n$ if $|I_v| = n$. A blow-up is *balanced* if the independent sets $(I_v)_{v\in G}$ are as equal in size as possible. We use $G(t)$ to denote the graph obtained by blowing-up each vertex of $G$ by $t$; that is, $G(t)$ is the balanced blow-up of $G$ on $t|G|$ vertices. As an example, the balanced blow-up of the $r$-clique, $K_r$, on $n$ vertices is exactly the Turán graph $T_r(n)$.

We note in passing that a graph has the same chromatic and clique number as any of its blow-ups. Furthermore, the chromatic number of a balanced blow-up of a graph is very robust – it cannot be changed by the deletion of a few edges.

**Lemma 1.7** (balanced blow-ups have robust chromatic number)**.** *Let $H$ be a graph that is not $r$-colourable and let $G$ be a balanced blow-up of $H$ on $n$ vertices. Then $G$ requires the deletion of $\Omega(n^2)$ edges to become $r$-colourable.*

*Proof.* Let the sizes of the $|H|$ parts of $G$ be $n_1, n_2, \dots, n_{|H|}$ where

$$\lceil n/|H|\rceil \geqslant n_1 \geqslant n_2 \geqslant \cdots \geqslant n_{|H|} \geqslant \lfloor n/|H|\rfloor.$$

Making $G$ $r$-colourable requires no copies of $H$ remaining. Now $G$ contains $n_1 n_2 \cdots n_{|H|}$ copies of $H$ in which each vertex is in the corresponding part, and each edge of $G$ is in at most $n_1 n_2 \cdots n_{|H|-2}$ such copies. Hence, to remove all copies of $H$ requires the deletion of at least $n_{|H|-1} n_{|H|} \geqslant \lfloor n/|H| \rfloor^2 = \Omega(n^2)$ edges. $\square$

Note that two vertices in the same $I_v$ have identical neighbourhoods. More generally, we term two vertices of any graph with the same neighbourhood *twins* – it immediately follows from this definition that twins are not adjacent.

One can view blowing-up a vertex by $n$ as weighting that vertex by $n$. A *weighted graph* $(G, \omega)$ is a graph $G$ together with a weighting $\omega\colon V(G) \to \mathbb{Q}^+$. When all the weights are integers one can view $(G, \omega)$ as a blow-up of $G$ in which each vertex $v$ has been blown-up by $\omega(v)$. Of course, if there are some non-integer (but rational) weights, scaling everything gives a suitable integer weighting. A normal (unweighted) graph is just a weighted graph in which every vertex has weight one. With this identification one has the natural notion of the order of a weighted graph $(G, \omega)$ and the degrees of its vertices. These are

$$\begin{aligned} \omega(G) &= \sum_{v \in V(G)} \omega(v), \\ d(v, \omega) &= \sum_{u\colon uv \in E(G)} \omega(u). \end{aligned}$$

A further notion related to blow-ups is that of graph homomorphisms. A *homomorphism from $G$ to $H$* is a function $\varphi\colon V(G) \to V(H)$ such that, for every edge $uv$ of $G$, $\varphi(u)\varphi(v)$ is an edge of $H$. We say *$G$ is homomorphic to $H$*, written $G \to H$, if there is a homomorphism from $G$ to $H$. Note that $G$ is homomorphic to $H$ if and only if $G$ is a subgraph of some blow-up of $H$. In particular, if $G \to H$, then $\chi(G) \leqslant \chi(H)$.

We say that a family of graphs $\mathcal{F}$ is *closed under blow-ups* if for every $G \in \mathcal{F}$ and every blow-up $G'$ of $G$ we have $G' \in \mathcal{F}$ also. As an example, the family of triangle-free graphs is closed under blow-ups (since the clique number is preserved), while the family of $C_5$-free graphs is not: $K_3$ does not contain a 5-cycle, but $K_3(2)$ does. In light of the previous paragraph we expect such families to behave well with homomorphisms. Suppose we have a family $\mathcal{F}$ that is both monotone and closed under taking blow-ups. For $G \in \mathcal{F}$ and $H \to G$, as $H$ is a subgraph of some blow-up of $G$, we must have $H \in \mathcal{F}$ also. The converse is also true (if the property holds, then $\mathcal{F}$ must be both monotone and closed under blow-ups). In particular, the family of graphs to which the 5-cycle (or any other graph) is not homomorphic is closed under blow-ups. Families that are closed under blow-ups are particularly well-behaved, a phenomenon we will meditate further on in § 5.1.1.

Finally, let us define some of the notation that pervades this thesis. More chapter-specific notation is defined in the corresponding chapter.

If $v$ is a vertex of a graph $G$, then $\Gamma(v) = \{u \in G \colon uv \in E(G)\}$ is the *neighbourhood of* $v$. The *degree of* $v$ is the size of $\Gamma(v)$ and is denoted by $\deg(v)$ or, more succinctly, $d(v)$. We write $G_v$ for $G[\Gamma(v)]$, the graph induced by the neighbourhood of $v$. More generally, for a set of vertices $X \subset G$, we write $\Gamma(X)$ for $\bigcap_{v\in X} \Gamma(v)$ (the common neighbourhood of the vertices of $X$), $\deg(X)$ or $d(X)$ for $|\Gamma(X)|$, and $G_X$ for $G[\Gamma(X)]$. We often omit set parentheses, so $\Gamma(u,v) = \Gamma(u) \cap \Gamma(v)$, $d(u,v) = |\Gamma(u,v)|$, and $G_{u,v}$ denote respectively the common neighbours of $u$ and $v$, their number, and the graph induced by them.

Given two graphs $G$ and $H$, the *join* of $G$ and $H$, denoted by $G + H$, is the graph obtained by taking disjoint copies of $G$ and $H$ and joining each vertex of the copy of $G$ to each vertex of the copy of $H$. Note that the chromatic and clique numbers of $G + H$ are the sum of the chromatic and clique numbers of $G$ and $H$.

Finally, we will often use asymptotic notation. For functions $f$ and $g$ from the real numbers, $\mathbb{R}$, or the positive integers, $\mathbb{Z}^+$, to the reals, we write the following:

- $f = o(g)$ if $\frac{f(n)}{g(n)} \to 0$ as $n \to \infty$,
- $f = \mathcal{O}(g)$ if there is a $C > 0$ such that, for all sufficiently large $n$, $|f(n)| \leqslant C|g(n)|$,
- $f = \omega(g)$ if $g = o(f)$, that is, if $\frac{f(n)}{g(n)} \to \infty$ as $n \to \infty$,
- $f = \Omega(g)$ if $g = \mathcal{O}(f)$, and
- $f = \Theta(g)$ if both $f = \mathcal{O}(g)$ and $g = \mathcal{O}(f)$.

# Chapter 2

# Graphs with no Induced $K_{2,t}$

Some results of this chapter have been published in [Ill21b] and some have been submitted in a forthcoming paper [Ill21a].

## 2.1 Introduction

Erdős and Simonovits's results [Erd67a; Erd68; Sim68], summarised in Theorem 1.1, solve to leading order the classical Turán problem for $H$-free graphs: the extremal graphs are $T_{\chi(H)-1}(n)$ with a smattering of edges added and deleted. The Turán problem for no induced $H$ is simple: if $H$ is a clique, then it is the same as for the normal subgraph problem (and so is answered by Turán's theorem [Tur41]) and if not, then the maximum number of edges is plainly $\binom{n}{2}$. That is,

$$\mathrm{ex}(n, H\text{-ind}) = \begin{cases} e(T_r(n)) & \text{if } H = K_{r+1}, \\ \binom{n}{2} & \text{if } H \text{ is not a clique.} \end{cases}$$

However, natural and interesting induced questions remain. One, recently introduced by Loh, Tait, Timmons, and Zhou [LTTZ18], simultaneously forbids an induced copy of one graph as well as a (not necessarily induced) copy of another. Forbidding a (non-induced) subgraph removes the possibility of the complete graph being extremal and so dismisses the humdrum answer $\binom{n}{2}$. They defined the *induced Turán number*

$$\mathrm{ex}(n, \{H, F\text{-ind}\})$$

to be the maximum number of edges in an $n$-vertex graph with no copy of $H$ and no induced copy of $F$. This induced Turán problem can be viewed as a generalisation of the *Ramsey-Turán problem*. Introduced by Sós [Sós69], the Ramsey-Turán number $\mathrm{RT}(n, H, m)$ is the greatest number of edges in an $n$-vertex graph with no independent

$m$-set and no copy of $H$. This is, of course, the same as the induced Turán problem when $F$ is an independent set on $m$ vertices. Ramsey-Turán theory has a long and rich history – see, for example, the survey of Simonovits and Sós [SS01].

In § 2.2, we asymptotically determine the induced Turán number except when $H$ is bipartite, or $F$ is an independent set or a complete bipartite graph.

**Theorem 2.1.** *Let $H$ and $F$ be graphs with chromatic numbers $r+1$ and $k+1$, respectively. Then*

$$\mathrm{ex}(n, \{H, F\text{-ind}\}) = \begin{cases} \left(1 - \frac{1}{r} + o(1)\right)\binom{n}{2} & \text{if } k \geqslant r \text{ or } F \text{ is not complete multipartite,} \\ \left(1 - \frac{1}{k} + o(1)\right)\binom{n}{2} & \text{if } k < r \text{ and } F \text{ is complete multipartite.} \end{cases}$$

Note that if $F$ is either an independent set or a complete bipartite graph, then Turán-style graphs are ruled out, and so, in this case, we expect the induced Turán number $\mathrm{ex}(n, \{H, F\text{-ind}\})$ to differ greatly from the usual Turán number $\mathrm{ex}(n, H)$. Loh, Tait, Timmons, and Zhou gave upper bounds for complete bipartite $F$, showing that

$$\mathrm{ex}(n, \{K_{r+1}, K_{s,t}\text{-ind}\}) = \mathcal{O}_r\left(n^{2-1/s}\right),$$

which recovers the same asymptotic growth rate as the Kővári-Sós-Turán theorem for graphs with no $K_{s,t}$ – this is far from the realm of classical non-degenerate Turán problems. They particularly focussed on the case $s = 2$, producing the sharper bound

$$\mathrm{ex}(n, \{K_{r+1}, K_{2,t}\text{-ind}\}) < (\sqrt{2} + o(1))(t-1)^{1/2}(r+t)^{(t-1)/2} n^{3/2}. \tag{2.1}$$

They also noted that if $H$ is not bipartite, then any bipartite subgraph of a $K_{2,t}$-free graph is $H$-free and contains no induced $K_{2,t}$. Thus

$$\mathrm{ex}(n, \{H, K_{2,t}\text{-ind}\}) \geqslant \tfrac{1}{2}\,\mathrm{ex}(n, K_{2,t}) \geqslant \tfrac{1}{4}(t-1)^{1/2} n^{3/2} - \mathcal{O}_t\left(n^{4/3}\right)$$

where the final inequality follows from a construction of Füredi [Für96]. In particular, for non-bipartite $H$, $\mathrm{ex}(n, \{H, K_{2,t}\text{-ind}\}) = \Theta_t(n^{3/2})$ but with a large gap for the growth rate in $t$.

The main result of this chapter, Theorem 2.9, gives an improved upper bound on the number of edges in an $n$-vertex $H$-free graph with no induced $K_{2,t}$. It is applicable in a wide variety of contexts, including both when $H$ is fixed but also when it is allowed to grow with $n$. For cleanliness, we will only state our results for complete $H$ in this introduction but will prove general results in subsequent sections.

We normalise the problem as follows. Fix an integer $t \geqslant 2$ and consider a graph $G$

on $n$ vertices with $\alpha\binom{n}{2}$ edges which does not contain an induced $K_{2,t}$. Our question is how large need $\alpha$ be to ensure that $G$ contains some subgraph $H$? We consider two regimes: $\alpha$ bounded away from zero, corresponding to $H$ growing with $n$, and $\alpha = o_n(1)$, corresponding to $H$ being fixed.

First consider when $\alpha$ is bounded away from zero, and so $G$ contains large substructures that grow with $n$. As an example, $G$'s clique number, $\omega(G)$, will go to infinity. Gyárfás, Hubenko, and Solymosi [GHS02] gave a lower bound for the clique number in the case when $t = 2$ (that is, $G$ contains no induced $C_4$), confirming a conjecture of Erdős.

**Theorem 2.2** (Gyárfás-Hubenko-Solymosi). *Let $G$ be a graph on $n$ vertices with $\alpha\binom{n}{2}$ edges. If $G$ does not contain an induced $K_{2,2}$, then $\omega(G) \geqslant \alpha^2 n/10$.*

This was recently improved by Holmsen [Hol20] (note $1 - \sqrt{1-\alpha} \geqslant \alpha/2$ for $\alpha \in [0, 1]$).

**Theorem 2.3** (Holmsen). *Let $G$ be a graph on $n$ vertices with $\alpha\binom{n}{2}$ edges. If $G$ does not contain an induced $K_{2,2}$, then $\omega(G) \geqslant (1 - \sqrt{1-\alpha})^2 n$.*

This result has the added advantage that $(1 - \sqrt{1-\alpha})^2 \to 1$ as $\alpha \to 1$, so it is approximately tight in this case. Using Theorem 2.9, we extend Theorems 2.2 and 2.3 to the general case of not containing an induced $K_{2,t}$. This extension is Theorem 2.11, which gives the correct growth rate in $n$ and furthermore, we believe is tight as $\alpha \to 1$ – see Remark 2.12. The following corollaries of Theorem 2.11 are derived in § 2.4. The $t = 3$ case is particularly clean.

**Theorem 2.4.** *Let $G$ be a graph on $n$ vertices with $\alpha\binom{n}{2}$ edges. If $G$ does not contain an induced $K_{2,3}$, then*

$$\omega(G) \geqslant \left\lfloor \tfrac{2}{3}\alpha\sqrt{n} \right\rfloor \text{ for all } n, \text{ and}$$
$$\omega(G) \geqslant \tfrac{1}{3}\alpha\sqrt{n\log n} + 2 \text{ for large enough } n \text{ in terms of } \alpha.$$

**Theorem 2.5.** *Let $G$ be a graph on $n$ vertices with $\alpha\binom{n}{2}$ edges. If $G$ does not contain an induced $K_{2,t}$, then*

$$\omega(G) \geqslant \left\lfloor \tfrac{t-1}{4}(\alpha^2 n)^{\frac{1}{t-1}} \right\rfloor - t + 3 \text{ for all } n, \text{ and}$$
$$\omega(G) \geqslant \tfrac{1}{20t}\left(\alpha^2 n(\log n)^{t-2}\right)^{\frac{1}{t-1}} \text{ for large enough } n \text{ in terms of } \alpha.$$

These improve the previous best lower bounds known due to Loh, Tait, Timmons, and Zhou [LTTZ18, Theorem 1.3]. The dependence upon $t$ has been improved for all $\alpha$ and a $(\log n)^{1-1/(t-1)}$ factor has been added for constant $\alpha > 0$.

We now return to the induced Turán numbers and the case where $\alpha = o_n(1)$. Nikiforov, Tait, and Timmons [NTT18] gave spectral versions of Loh, Tait, Timmons, and Zhou's results, obtaining the following version of inequality (2.1)

$$\mathrm{ex}(n, \{K_{r+1}, K_{2,t}\text{-ind}\}) < \tfrac{1}{2}R(t,r)n^{3/2}.$$

It is natural for Ramsey numbers to appear as the class of graphs with 'no induced $K_{2,t}$' includes those with 'no independent $t$-set'. We improve the dependence upon $t$ and $r$, replacing the Ramsey number by its square root.

**Theorem 2.6.** *For any integers $t \geqslant 2$ and $r \geqslant 1$,*

$$\begin{aligned}\mathrm{ex}(n, \{K_{r+1}, K_{2,t}\text{-ind}\}) &< \tfrac{t}{2\sqrt{t-1}}R(t,r)^{1/2}n^{3/2},\\ \mathrm{ex}(n, \{K_{r+1}, K_{2,t}\text{-ind}\}) &\leqslant \tfrac{1}{2}\big(R(t,r) - 1 + o(1)\big)^{1/2}n^{3/2}.\end{aligned}$$

The second of these is a sharper, although asymptotic, bound. An intermediate result in its proof, Theorem 2.17, says that any $K_{r+1}$-free graph with no induced $K_{2,t}$ has at most $\mathcal{O}_{t,r}(n^{27/14}) = o(n^2)$ triangles. This seems of independent interest: any graph with bounded clique number and no induced $K_{2,t}$ has $o(n^2)$ triangles.

Our upper bounds for $\mathrm{ex}(n, \{H, K_{2,t}\text{-ind}\})$, Theorems 2.16 and 2.18, involve Ramsey numbers for $H$. For $H$ with good Ramsey bounds, much sharper upper bounds than those in Theorem 2.6 are, of course, possible.

When $H$ is an odd cycle, Ergemlidze, Győri, and Methuku [EGM19] obtained tight asymptotic bounds. They showed that

$$\mathrm{ex}(n, \{C_{2k+1}, K_{2,t}\text{-ind}\}) = (1 + o(1))(t-1)^{1/2}\left(\tfrac{n}{2}\right)^{3/2},$$

except when $k = t = 2$. They bounded $\mathrm{ex}(n, \{C_5, C_4\text{-ind}\})$ between $(2/\sqrt{27} + o(1))n^{3/2}$ and $(1/2 + o(1))n^{3/2}$.

## 2.2 General Induced Turán Numbers

Here we prove Theorem 2.1 determining to leading order the induced Turán number for non-bipartite $H$ and $F$ not independent nor complete bipartite.

**Theorem 2.1.** *Let $H$ and $F$ be graphs with chromatic numbers $r + 1$ and $k + 1$, respectively.*

*Then*

$$\mathrm{ex}(n, \{H, F\text{-ind}\}) = \begin{cases} \left(1 - \frac{1}{r} + o(1)\right)\binom{n}{2} & \text{if } k \geqslant r \text{ or } F \text{ is not complete multipartite,} \\ \left(1 - \frac{1}{k} + o(1)\right)\binom{n}{2} & \text{if } k < r \text{ and } F \text{ is complete multipartite.} \end{cases}$$

*Proof.* If $k \geqslant r$ or $F$ is not complete multipartite, then $F$ is not an induced subgraph of the Turán graph $T_r(n)$. Hence,

$$e(T_r(n)) \leqslant \mathrm{ex}(n, \{H, F\text{-ind}\}) \leqslant \mathrm{ex}(n, H).$$

Theorem 1.1 and the lower bound $(1 - 1/r)\binom{n}{2}$ for $e(T_r(n))$ give the first half of the theorem.

Now suppose that $k < r$ and $F$ is complete $(k+1)$-partite. Firstly the Turán graph $T_k(n)$ contains neither $H$ nor $F$ as subgraphs (let alone induced ones) so,

$$\mathrm{ex}(n, \{H, F\text{-ind}\}) \geqslant e(T_k(n)) \geqslant \left(1 - \tfrac{1}{k}\right)\binom{n}{2}.$$

As $F$ is complete $(k+1)$-partite, there is a positive integer $t$ such that $F$ is an induced subgraph of $K_{k+1}(t)$. By Ramsey's theorem [Ram30], there is a positive integer $s$ such that any $s$-vertex graph contains either an independent $t$-set or a copy of $H$.

Fix $\varepsilon > 0$ and let $G$ be an $n$-vertex graph with at least $(1 - 1/k + \varepsilon)\binom{n}{2}$ edges where $n$ is sufficiently large. By the Erdős-Stone theorem [ES46] (or Theorem 1.1), $G$ contains a copy of $K_{k+1}(s)$. Let the parts of the $K_{k+1}(s)$ be $V_1, \dots, V_{k+1}$ so each one has $s$ vertices. If any $V_i$ contains $H$, then $G$ does. Otherwise, by the definition of $s$, each $V_i$ contains an independent set of size $t$. Thus $G$ contains an induced copy of $K_{k+1}(t)$ and so an induced copy of $F$. Thus, for all large $n$,

$$\mathrm{ex}(n, \{H, F\text{-ind}\}) \leqslant \left(1 - \tfrac{1}{k} + \varepsilon\right)\binom{n}{2},$$

as required. □

**Remark 2.7.** If $H$ is bipartite and $F$ is not, then the Turán number $\mathrm{ex}(n, H)$ and induced Turán number $\mathrm{ex}(n, \{H, F\text{-ind}\})$ are within a factor of two. Indeed, any bipartite subgraph of an extremal $H$-free graph is $F$-free. When $H$ and $F$ are both bipartite, the situation is unclear – it seems likely to depend upon the fine structure of both graphs.

## 2.3 Notation and Main Result

For a fixed graph $H$, let $\{H-x\}$ be the set of graphs obtained by removing a single vertex from $H$ and let $\{H-\bar{e}\}$ be the set of graphs obtained from $H$ by either removing a single vertex or two non-adjacent vertices. In particular, the Ramsey number $R(K_t, \{H-x\})$ is the least $n$ such that any red-blue colouring of the edges of $K_n$ contains either a red $K_t$ or a blue graph that can be obtained from $H$ by removing a single vertex.

To state our main result it will be convenient to first define a constant $\beta$ depending upon $\alpha$ and $t$.

**Definition 2.8.** *Given $\alpha \in [0,1]$ and an integer $t \geqslant 2$, define*

$$\beta_t(\alpha) = \frac{t}{2\sqrt{t-1}}\left[\sqrt{1-\left(1-\tfrac{2}{t}\right)^2\alpha} - \sqrt{1-\alpha}\right].$$

Note that $\beta_2(\alpha) = 1-\sqrt{1-\alpha}$ so Theorem 2.3 can be rephrased: if $G$ is a graph on $n$ vertices with $\alpha\binom{n}{2}$ edges containing no induced $K_{2,2}$, then $\omega(G) \geqslant \beta_2(\alpha)^2 n$.

Our main result is the following which applies for all values of $\alpha$.

**Theorem 2.9.** *Fix a graph $H$. Let $G$ be a graph on $n$ vertices with $\alpha\binom{n}{2}$ edges containing no induced $K_{2,t}$ $(t \geqslant 2)$ and let $\beta = \beta_t(\alpha)$.*

*If $R(K_t, \{H-x\}) \leqslant \beta^2 n$, then $H$ is a subgraph of $G$. In particular, if $R(K_t, \{H-x\}) \leqslant \frac{t-1}{t^2}\cdot\alpha^2 n$, then $H$ is a subgraph of $G$.*

The sufficiency of $R(K_t, \{H-x\}) \leqslant \frac{t-1}{t^2}\cdot\alpha^2 n$ follows from the following lemma which relates $\beta$ to $\alpha$ in a manageable way.

**Lemma 2.10.** *For all $\alpha \in [0,1]$ and integers $t \geqslant 2$, $\beta = \beta_t(\alpha)$ satisfies*

$$(t-1)\left(\alpha-\beta^2\right)^2 = t^2(1-\alpha)\beta^2, \tag{2.2}$$

$$\tfrac{\sqrt{t-1}}{t}\alpha \leqslant \beta \leqslant \alpha,$$

$$\beta \to 1, \text{ as } \alpha \to 1.$$

*Proof.* Equation (2.2) is a quadratic in $\beta^2$. Writing $\gamma$ for $\beta^2$ and $u$ for $(t-1)/t^2$ and dividing by $t^2$ gives $u(\alpha-\gamma)^2 = (1-\alpha)\gamma$. This rearranges to

$$u\gamma^2 - \gamma(2u\alpha + 1 - \alpha) + u\alpha^2 = 0,$$

one of whose solutions is

$$\begin{aligned}\gamma &= \tfrac{1}{2u}\left[2u\alpha + 1 - \alpha - \sqrt{(2u\alpha+1-\alpha)^2 - 4u^2\alpha^2}\right]\\ &= \tfrac{1}{2u}\left[2u\alpha + 1 - \alpha - \sqrt{(1-\alpha)(4u\alpha+1-\alpha)}\right]\\ &= \tfrac{1}{4u}\left[(1-\alpha) + (1-(1-4u)\alpha) - 2\sqrt{(1-\alpha)(1-(1-4u)\alpha)}\right].\end{aligned}$$

In particular,

$$\beta = \tfrac{1}{2\sqrt{u}}\left[\sqrt{1-(1-4u)\alpha} - \sqrt{1-\alpha}\right]$$

satisfies equation (2.2). Note that $1-4u = (1-2/t)^2$ and so $\beta_t(\alpha)$ does indeed satisfy equation (2.2).

Fix $t$ and define the function $f(x) = \sqrt{1-(1-2/t)^2x} - \sqrt{1-x}$ for $x \in [0,1]$. Then $f$ is convex increasing with $f(0) = 0$ and $f(1) = \frac{2\sqrt{t-1}}{t}$. Thus $f(x) \leqslant \frac{2\sqrt{t-1}}{t}x$. Also the derivative of $f$ at zero is $\frac{2}{t} - \frac{2}{t^2} = \frac{2(t-1)}{t^2}$ so $f(x) \geqslant \frac{2(t-1)}{t^2}x$. In particular, $\beta = \frac{t}{2\sqrt{t-1}}f(\alpha)$ satisfies $\frac{\sqrt{t-1}}{t}\alpha \leqslant \beta \leqslant \alpha$.

Finally, $f$ is continuous so, as $\alpha$ tends to 1, $\beta$ tends to $\frac{t}{2\sqrt{t-1}}f(1) = 1$. □

We prove Theorem 2.9 in § 2.5. Before that, in § 2.4, we give the promised lower bounds, Theorems 2.4 and 2.5, for the clique number of graphs with no induced $K_{2,t}$. Finally, in § 2.6, we prove upper bounds for $\mathrm{ex}(n, \{H, K_{2,t}\text{-ind}\})$.

## 2.4 Clique Numbers of Graphs with no Induced $K_{2,t}$

If we take $H = K_{r+1}$ in Theorem 2.9, then $\{H - x\} = \{K_r\}$, so we immediately obtain the following. As $R(2,r) = r$, this recovers Holmsen's result, Theorem 2.3.

**Theorem 2.11.** *Let $G$ be a graph on $n$ vertices with $\alpha\binom{n}{2}$ edges containing no induced $K_{2,t}$ and let $\beta = \beta_t(\alpha)$. For any positive integer $r$ with $R(t,r) \leqslant \beta^2 n$, we have $\omega(G) \geqslant r+1$.*

**Remark 2.12.** Consider a graph $G$ on $n$ vertices which has no independent $t$-set and smallest possible clique number (a Ramsey-like graph): that is $R(t, \omega(G)+1) > n \geqslant R(t, \omega(G))$. Now $G$ has no independent $t$-set so does not contain an induced $K_{2,t}$. This shows that if $\beta$ is replaced by 1, then the result no longer holds. In particular, Theorem 2.11 gives the correct growth rate for $\omega(G)$ in terms of $n$, when $\alpha$ is bounded away from zero. Furthermore, if there are such $G$ with $(1-o(1))\binom{n}{2}$ edges, then these form a sequence of graphs for which $\alpha \to 1$ (and so $\beta \to 1$). It is for this reason we believe the result is in a sense tight as $\alpha \to 1$.

We do believe that such graphs have $(1-o(1))\binom{n}{2}$ edges. This would follow, for example, from $\frac{R(t-1,m)}{R(t,m)} \to 0$ as $m \to \infty$ (true for $t = 3$ and 4 by standard Ramsey bounds – say those found in [Bol01] – but not known in general): the non-neighbours of a vertex in such a graph $G$ cannot contain an independent $(t-1)$-set, so there are at most $R(t-1, \omega(G)+1)$ non-neighbours, and so $\delta(G)$ would be $(1-o(1))n$.

The following corollary for $t = 3$ contains Theorem 2.4.

**Corollary 2.13.** *Let $G$ be a graph on $n$ vertices with $\alpha\binom{n}{2}$ edges which contains no induced $K_{2,3}$. Let $\beta = \beta_3(\alpha) = \frac{3}{2\sqrt{2}}\left[\sqrt{1-\alpha/9} - \sqrt{1-\alpha}\right]$. Then*

$$\begin{aligned}
&\omega(G) \geqslant \lfloor \beta\sqrt{2n} \rfloor \geqslant \lfloor \tfrac{2}{3}\alpha\sqrt{n} \rfloor \text{ for all } n, \text{ and}\\
&\omega(G) \geqslant \beta\sqrt{n\log n/2} + 2 \geqslant \tfrac{1}{3}\alpha\sqrt{n\log n} + 2 \text{ for large enough } n, \text{ say } n \geqslant \exp(2e^2\beta^{-2}).
\end{aligned}$$

*Proof.* Lemma 2.10 gives $\beta \geqslant \alpha\sqrt{2}/3$ so it suffices to prove the left-hand inequalities.

Firstly, the theorem of Erdős and Szekeres [ES35] gives $R(3,r) \leqslant \binom{r+1}{2}$ for all positive $r$. Thus $r = \lfloor \beta\sqrt{2n} \rfloor - 1$ satisfies $R(3,r) \leqslant \lfloor \beta\sqrt{2n} \rfloor^2/2 \leqslant \beta^2 n$ and so Theorem 2.11 gives the first result.

Secondly, $R(3,r) \leqslant \frac{(r-2)^2}{\log(r-1)-1}$ for all $r \geqslant 4$ (a corollary of Shearer's result on independent sets in triangle-free graphs, [She83]). Thus $r = \left\lfloor \beta\sqrt{n\log n/2} \right\rfloor + 2$ satisfies $R(3,r) \leqslant \beta^2 n$ provided $n \geqslant \exp(2e^2\beta^{-2})$. □

The following corollary, which contains Theorem 2.5, is obtained in exactly the same way, using known bounds for $R(t,r)$. Improvements in the upper bounds on Ramsey numbers would improve the results.

**Corollary 2.14.** *Let $G$ be a graph on $n$ vertices with $\alpha\binom{n}{2}$ edges containing no induced $K_{2,t}$ and let $\beta = \beta_t(\alpha)$. Then*

$$\begin{aligned}
&\omega(G) \geqslant \left\lfloor \tfrac{t-1}{e}(\beta^2 n)^{\frac{1}{t-1}} \right\rfloor - t + 3 \text{ and } \omega(G) \geqslant \left\lfloor \tfrac{t-1}{4}(\alpha^2 n)^{\frac{1}{t-1}} \right\rfloor - t + 3 \text{ for all } n, \text{ and}\\
&\omega(G) \geqslant \tfrac{1}{20}\left(\beta^2 n\right)^{\frac{1}{t-1}}\left(\tfrac{\log n}{t-1}\right)^{1-\frac{1}{t-1}} \geqslant \tfrac{1}{20t}\left(\alpha^2 n(\log n)^{t-2}\right)^{\frac{1}{t-1}} \text{ for large enough } n \text{ in terms of } \beta.
\end{aligned}$$

*Proof.* The theorem of Erdős and Szekeres [ES35] gives that $R(t,r) \leqslant \binom{r+t-2}{t-1} \leqslant \frac{(r+t-2)^{t-1}}{(t-1)!}$ for all positive $r$. Thus $r = \left\lfloor \left(\beta^2 n(t-1)!\right)^{\frac{1}{t-1}} \right\rfloor - t + 2$ has $R(t,r) \leqslant \beta^2 n$ so, by Theorem 2.11,

$$\omega(G) \geqslant \left\lfloor \left(\beta^2 n(t-1)!\right)^{\frac{1}{t-1}} \right\rfloor - t + 3 \geqslant \left\lfloor \left(\tfrac{t-1}{t^2}\alpha^2 n(t-1)!\right)^{\frac{1}{t-1}} \right\rfloor - t + 3.$$

Furthermore $(t-1)! \geqslant \left(\frac{t-1}{e}\right)^{t-1}$ so $\left((t-1)!\right)^{\frac{1}{t-1}} \geqslant \frac{t-1}{e}$. That $\left(\frac{t-1}{t^2}(t-1)!\right)^{\frac{1}{t-1}} \geqslant \frac{t-1}{4}$ follows from $(t-1)! \geqslant \frac{(t-1)^{t-1/2}}{e^{t-1}}$ for $t \geqslant 4$ and can be checked directly for $t = 2, 3$.

Finally $R(t,r) \leqslant 2(20)^{t-3}\frac{r^{t-1}}{(\log r)^{t-2}}$ for $r$ sufficiently large (see Bollobás [Bol01, Thm 12.17]) so we obtain, for all large $n$, that

$$\omega(G) \geqslant \frac{1}{20}\left(\frac{\beta^2 n(\log n)^{t-2}}{(t-1)^{t-2}}\right)^{\frac{1}{t-1}} \geqslant \frac{1}{20}\left(\frac{\alpha^2 n(\log n)^{t-2}}{t^2(t-1)^{t-3}}\right)^{\frac{1}{t-1}}. \qquad \square$$

## 2.5 Proof of Main Result

For convenience we restate the main result here. Some ideas of the proof are inspired by Holmsen [Hol20].

**Theorem 2.9.** *Fix a graph $H$. Let $G$ be a graph on $n$ vertices with $\alpha\binom{n}{2}$ edges containing no induced $K_{2,t}$ $(t \geqslant 2)$ and let $\beta = \beta_t(\alpha)$.*

*If $R(K_t, \{H-x\}) \leqslant \beta^2 n$, then $H$ is a subgraph of $G$. In particular, if $R(K_t, \{H-x\}) \leqslant \frac{t-1}{t^2} \cdot \alpha^2 n$, then $H$ is a subgraph of $G$.*

*Proof.* By Lemma 2.10, for $\alpha \in [0,1]$ we have $0 \leqslant \beta \leqslant \alpha \leqslant 1$ and also $\frac{t-1}{t^2}\left(\alpha - \beta^2\right)^2 = (1-\alpha)\beta^2$.

Suppose that $G$ does not contain $H$. Let the set of missing edges in $G$ be $M = \binom{V(G)}{2} - E(G)$, which has size $(1-\alpha)\binom{n}{2}$. For each $v \in V(G)$, let

$m_v$ be the total number of missing edges in $G_v$,
$\bar{\Delta}_1, \ldots, \bar{\Delta}_{\gamma_v}$ be a maximal collection of pairwise vertex-disjoint independent $t$-sets in $G_v$.

By the maximality of $\gamma_v$, $G[\Gamma(v) \setminus \cup_j \bar{\Delta}_j]$ does not contain an independent $t$-set. Furthermore it does not contain any $H - x$ (else together with $v$ we have a copy of $H$ in $G$). Thus

$$R(K_t, \{H-x\}) - 1 \geqslant |\Gamma(v)| - t\gamma_v = \deg(v) - t\gamma_v,$$

and so

$$\gamma_v \geqslant \tfrac{1}{t}[\deg(v) - R(K_t, \{H-x\}) + 1] \geqslant \tfrac{1}{t}[\deg(v) - \beta^2(n-1)]. \tag{2.3}$$

$G$ contains no induced $K_{2,t}$ so at most one vertex in $\bar{\Delta}_i$ is adjacent to all of $\bar{\Delta}_j$ (for any $i \neq j$). In particular, between $\bar{\Delta}_i$ and $\bar{\Delta}_j$ there must be at least $t-1$ missing edges. These missing edges are in no $\bar{\Delta}_k$ (by vertex-disjointness) and each such edge corresponds to only one pair $(\bar{\Delta}_i, \bar{\Delta}_j)$. Considering these missing edges as well as the ones contained entirely in each $\bar{\Delta}_k$ gives

$$m_v \geqslant \tbinom{t}{2}\gamma_v + (t-1)\tbinom{\gamma_v}{2} = q(\gamma_v),$$

where

$$q(x) = \tfrac{t-1}{2} \cdot x(x+t-1)$$

is convex and increasing for non-negative $x$. Averaging (2.3) over $v \in G$ we have

$$\frac{1}{n}\sum_{v \in G} \gamma_v \geqslant \tfrac{1}{tn}[2e(G) - \beta^2 n(n-1)] = \tfrac{1}{t}(\alpha - \beta^2)(n-1).$$

Using Jensen, the monotonicity of $q$, and the fact that $\alpha \geqslant \beta \geqslant \beta^2$ gives

$$\begin{aligned}
\frac{1}{n}\sum_{v \in G} m_v \geqslant \frac{1}{n}\sum_{v \in G} q(\gamma_v) \geqslant q\left(\frac{1}{n}\sum_{v \in G}\gamma_v\right) &\geqslant q\left(\tfrac{1}{t}(\alpha - \beta^2)(n-1)\right) \\
&= \tfrac{t-1}{2} \cdot \tfrac{1}{t}(\alpha - \beta^2)(n-1) \cdot \left(\tfrac{1}{t}(\alpha - \beta^2)(n-1) + t - 1\right) \\
&\geqslant \tfrac{t-1}{2} \cdot \tfrac{1}{t}(\alpha - \beta^2)(n-1) \cdot \tfrac{1}{t}(\alpha - \beta^2)n \\
&= \tfrac{t-1}{t^2}(\alpha - \beta^2)^2 \cdot \tbinom{n}{2} \\
&= \beta^2(1-\alpha)\tbinom{n}{2}.
\end{aligned}$$

Now $\sum_{v \in G} m_v = \sum_{\bar{e} \in M} |\{v \colon \bar{e} \subset \Gamma(v)\}|$ and $|M| = (1-\alpha)\binom{n}{2}$ so there is $\bar{e} \in M$ contained in the neighbourhood of at least $\beta^2 n$ vertices. Let $S \subset V(G)$ be a set of at least $\beta^2 n$ vertices such that $\bar{e} \subset \Gamma(v)$ for all $v \in S$.

Now $G[S]$ contains no independent $t$-set (else together with $\bar{e}$ we have an induced $K_{2,t}$) and $|S| \geqslant \beta^2 n \geqslant R(K_t, \{H - x\})$ so $G[S]$ contains a copy of some $H - x$. Together with one end-vertex of $\bar{e}$ we have a copy of $H$ in $G$. □

**Remark 2.15.** It is natural to ask whether the ideas of this argument could be extended to graphs with no induced $K_{s,t}$. The argument above is so clean partly because the number of independent 2-sets in $G$ is determined by $\alpha$ (it is $|M| = (1-\alpha)\binom{n}{2}$). Extending to no induced $K_{s,t}$ would require some knowledge of the number of independent $s$-sets in $G$.

## 2.6 Turán Number for no $H$ and no Induced $K_{2,t}$

We now focus on the regime where $\alpha$ goes to zero and obtain the upper bounds for the induced Turán numbers $\mathrm{ex}(n, \{H, K_{2,t}\text{-ind}\})$. Note that the first and second bounds of Theorem 2.6 follow readily from Theorems 2.16 and 2.18 respectively.

**Theorem 2.16.** *Fix a graph $H$. For any integer $t \geqslant 2$,*

$$\mathrm{ex}(n, \{H, K_{2,t}\text{-ind}\}) < \tfrac{t}{2\sqrt{t-1}} R(K_t, \{H - x\})^{1/2} n^{3/2}.$$

*Proof.* Let $G$ be a graph on $n$ vertices containing no induced $K_{2,t}$ and no copy of $H$. By Theorem 2.9, $R(K_t, \{H - x\}) > \frac{t-1}{t^2} \cdot \alpha^2 n$, so $\alpha < \frac{t}{\sqrt{t-1}} n^{-1/2} R(K_t, \{H - x\})^{1/2}$. Therefore,

$$e(G) = \alpha\tbinom{n}{2} < \tfrac{t}{2\sqrt{t-1}} R(K_t, \{H - x\})^{1/2} n^{1/2}(n-1). \qquad \square$$

We can improve this by first showing that $n$-vertex $H$-free graphs with no induced $K_{2,t}$ contain $o(n^2)$ triangles. This asymptotically improves our lower bound on the number of missing edges in each neighbourhood.

**Theorem 2.17.** *Fix a graph $H$ and an integer $t \geqslant 2$. Every $n$-vertex graph which contains no copy of $H$ and no induced $K_{2,t}$ has at most $\mathcal{O}(n^{27/14})$ triangles.*

*Proof.* By Theorem 2.16, there is a constant $C = C_{H,t}$ such that every $m$-vertex graph which contains no copy of $H$ and no induced $K_{2,t}$ has at most $Cm^{3/2}$ edges.

Let $G$ be a graph on $n$ vertices containing no induced $K_{2,t}$ and no copy of $H$. For each vertex $v$ of $G$, note that exactly $e(G_v)$ triangles in $G$ contain $v$. As $G$ has no copy of $H$ and no induced $K_{2,t}$,

$$\begin{aligned} e(G) &\leqslant Cn^{3/2}, \\ e(G_v) &\leqslant C \deg(v)^{3/2}. \end{aligned}$$

Let $X$ be the set of vertices in $G$ whose degree is at least $f(n)$ (a function of $n$ whose value we give later). Firstly,

$$|X| f(n) \leqslant \sum_{v \in X} \deg(v) \leqslant 2e(G) \leqslant 2Cn^{3/2},$$

and so the number of triangles in $G$ whose vertices are all in $X$ is at most

$$\tbinom{|X|}{3} \leqslant \tfrac{1}{6}|X|^3 \leqslant \tfrac{4}{3} C^3 n^{9/2} f(n)^{-3}.$$

The number of triangles of $G$ containing at least one vertex in $V(G) \setminus X$ is at most

$$\sum_{v \notin X} e(G_v) \leqslant C \sum_{v \notin X} \deg(v)^{3/2}.$$

The function $x \mapsto x^{3/2}$ is convex and all $v \notin X$ satisfy $\deg(v) \leqslant f(n)$, so

$$\sum_{v \notin X} \deg(v)^{3/2} \leqslant \left( f(n)^{-1} \sum_{v \notin X} \deg(v) \right) f(n)^{3/2} = f(n)^{1/2} \sum_{v \notin X} \deg(v)$$
$$\leqslant 2 f(n)^{1/2} e(G) \leqslant 2Cn^{3/2} f(n)^{1/2}.$$

Thus, the number of triangles in $G$ is at most

$$\tfrac{4}{3} C^3 n^{9/2} f(n)^{-3} + 2C^2 n^{3/2} f(n)^{1/2}.$$

We minimise this last expression by taking $f(n) = 2^{4/7} C^{2/7} n^{6/7}$ which gives a value less than $3C^{15/7} n^{27/14}$. □

**Theorem 2.18.** *Fix a graph $H$ and an integer $t \geqslant 2$. Let $\Delta(n, H, t)$ denote the greatest number of triangles in a graph on $n$ vertices containing no copy of $H$ and no induced $K_{2,t}$. Let $G$ be a graph on $n$ vertices with $\alpha\binom{n}{2}$ edges containing no induced $K_{2,t}$. If*

$$\alpha^2(n-1) > R(K_t, \{H - \bar{e}\}) - 1 + 3\Delta(n, H, t)\binom{n}{2}^{-1},$$

*then $H$ is a subgraph of $G$. In particular,*

$$\mathrm{ex}(n, \{H, K_{2,t}\text{-ind}\}) \leqslant \tfrac{1}{2}\big(R(K_t, \{H - \bar{e}\}) - 1 + o(1)\big)^{1/2} n^{3/2}.$$

*Proof.* $R(K_t, \{H - \bar{e}\}) \geqslant 2$ so we in fact have

$$\alpha[\alpha(n-1) - 1] > (1-\alpha)(R(K_t, \{H - \bar{e}\}) - 1) + 3\Delta(n, H, t)\binom{n}{2}^{-1}.$$

We will use this to show $H$ is a subgraph of $G$. Suppose for contradiction it is not. Let the set of missing edges in $G$ be $M = \binom{V(G)}{2} - E(G)$ which has size $(1-\alpha)\binom{n}{2}$. For each $v \in V(G)$ let

$$e_v = e(G_v),$$
$$m_v = \text{ total number of missing edges in } G_v.$$

First note that $e_v + m_v = \binom{|\Gamma(v)|}{2} = \binom{\deg(v)}{2}$, so, by Jensen's inequality,

$$\sum_{v \in G} (m_v + e_v) \geqslant n\binom{2e(G)/n}{2} = n\binom{\alpha(n-1)}{2} = \alpha[\alpha(n-1) - 1]\binom{n}{2}.$$

Now $e_v$ is also the number of triangles in $G$ containing $v$ so $\sum_{v \in G} e_v$ is three times the number of triangles in $G$ which is at most $3\Delta(n, H, t)$. Thus

$$\sum_{v \in G} m_v \geqslant \alpha[\alpha(n-1)-1]\binom{n}{2} - 3\Delta(n,H,t) > (1-\alpha)\binom{n}{2}(R(K_t, \{H-\bar{e}\})-1).$$

Now $\sum_{v \in G} m_v = \sum_{\bar{e} \in M} |\{v \colon \bar{e} \subset \Gamma(v)\}|$ and $|M| = (1-\alpha)\binom{n}{2}$ so there is some missing edge $\bar{e}$ and some $S \subset V(G)$ of size $R(K_t, \{H-\bar{e}\})$ with $\bar{e} \subset \Gamma(v)$ for each $v \in S$. $G[S]$ does not contain an independent $t$-set (else together with $\bar{e}$ we have an induced $K_{2,t}$ in $G$) so $G[S]$ contains a copy of some $H-x$ or some $H-\bar{e}$. Together with $\bar{e}$ we have that $G$ contains a copy of $H$ proving the first result.

By Theorem 2.17, $\Delta(n, H, t) = o(n^2)$. Suppose that $G$ is a graph on $n$ vertices with no $H$ and no induced $K_{2,t}$. We must have

$$\alpha^2(n-1) \leqslant R(K_t, \{H-\bar{e}\}) - 1 + o(1).$$

Using $e(G) = \alpha\binom{n}{2}$ gives the required result. □

# Chapter 3

# The Chromatic Profile of Locally Bipartite Graphs

Many of the results of this chapter have been submitted in a forthcoming paper [Ill20].

## 3.1 Introduction

This chapter and the next address the Erdős-Simonovits problem for some particularly natural families of graphs. We remind the reader that, for a family of graphs $\mathcal{F}$, the Erdős-Simonovits problem for $\mathcal{F}$ asks for upper bounds on the chromatic number of members of $\mathcal{F}$ with large minimum degree. In particular, it asks for the *chromatic profile of $\mathcal{F}$*: the sequence of values $\delta_\chi(\mathcal{F}, k)$ where

$$\delta_\chi(\mathcal{F}, k) = \inf\{d\colon \text{if } \delta(G) \geqslant d|G| \text{ and } G \in \mathcal{F}, \text{ then } \chi(G) \leqslant k\}.$$

This is a notion of the structure of graphs in $\mathcal{F}$ with large minimum degree. In the case where $\mathcal{F}$ is the family of $H$-free graphs, we write $\delta_\chi(H, k)$ for $\delta_\chi(\mathcal{F}, k)$ – determining this was the original question of Erdős and Simonovits [ES73]. They stated that for general $H$ this seemed 'too complicated' – a sentiment that was reiterated more recently by Allen, Böttcher, Griffiths, Kohayakawa, and Morris [ABG+13]. Indeed, for many $H$, even the chromatic number of the $H$-free graph with the most edges (let alone greatest minimum degree) is unknown.

The particular case of triangle-free graphs has been extensively studied. It has a rich history initiated by the paper of Erdős and Simonovits [ES73]. This conjectured that every $n$-vertex triangle-free graph with minimum degree greater than $n/3$ is 3-colourable and gave an example (constructed in collaboration with Hajnal) of $n$-vertex triangle-free graphs with minimum degree $(1/3 - o(1))n$ and arbitrarily large chromatic number. The example shows that $\delta_\chi(K_3,k) \geqslant 1/3$ for all $k$, while the conjecture claims $\delta_\chi(K_3,3) = 1/3$. A simple observation of Andrásfai [And62] (also the basic $r = 2$ version of the Andrásfai-Erdős-Sós theorem, Theorem 1.2 – see Lemma 5.9 for a proof) says that every $n$-vertex triangle-free graphs with minimum degree greater than $2/5 \cdot n$ is bipartite and so $\delta_\chi(K_3,2) \leqslant 2/5$. The 5-cycle shows there is equality. Next, Häggkvist [Häg82] produced a suitably weighted Grötzsch graph (displayed in Figure 3.4 on page 35) to disprove Erdős and Simonovits's conjecture. His graph was 4-chromatic, triangle-free, 10-regular and had 29 vertices implying $\delta_\chi(K_3,3) \geqslant 10/29$. He also showed that all $n$-vertex triangle-free graphs with minimum degree greater than $3/8 \cdot n$ are homomorphic to the 5-cycle and so $\delta_\chi(K_3,3) \leqslant 3/8$. Jin [Jin95] produced a mammoth argument to close the gap for $\delta_\chi(K_3,3)$ showing that it is in fact equal to 10/29. Building on these results and many others, Brandt and Thomassé [BT05] determined fully the chromatic profile of triangle-free graphs. They showed that every $n$-vertex triangle-free graph with minimum degree greater than $n/3$ is 4-colourable and so $\delta_\chi(K_3,k) = 1/3$ for all $k \geqslant 4$. Table 3.1 summarises this all concisely.

| $\delta(G)\lvert G\rvert^{-1} >$ | $2/5$ | $10/29$ | $1/3$ | $1/3-\varepsilon$ |
|---|---|---|---|---|
| $\chi(G) \leqslant$ | 2 | 3 | 4 | $\infty$ |

Table 3.1: Chromatic profile of a triangle-free graph $G$

Independently, Goddard and Lyle [GL10], and Nikiforov [Nik10] extended these using induction to determine the chromatic profile of $K_{r+1}$-free graphs (with $\delta_\chi(K_{r+1},r)$ already given by the Andrásfai-Erdős-Sós theorem).

Although the chromatic profile for general $H$ has proved out of reach, there has been much greater success with the *chromatic threshold*. The *chromatic threshold of a family* $\mathcal{F}$ is the limit (and so infimum) of the decreasing sequence $\delta_\chi(\mathcal{F},k)$: that is,

$$\begin{aligned}\delta_\chi(\mathcal{F}) &= \inf_k \delta_\chi(\mathcal{F},k)\\ &= \inf\{d\colon \exists C = C(\mathcal{F},d) \text{ such that if } \delta(G) \geqslant d\lvert G\rvert \text{ and } G \in \mathcal{F}, \text{then } \chi(G) \leqslant C\}.\end{aligned}$$

It essentially answers the question of what minimum degree guarantees graphs in $\mathcal{F}$ have bounded chromatic number. For triangle-free graphs, Erdős, Hajnal, and Simonovits's example shows that $\delta_\chi(K_3) \geqslant 1/3$. Thomassen [Tho02] was the first to prove the matching upper bound, although, of course, Brandt and Thomassé's result does so strongly. In a breakthrough paper, Allen et al. [ABG+13] determined the chromatic threshold of $H$-free graphs for each graph $H$. They also highlighted *locally bipartite graphs* as a natural family to consider.

**Definition 3.1.** *A graph is **locally bipartite** if the neighbourhood of every vertex induces a bipartite graph.*

*More generally, for a positive integer $r$, a graph is **locally $r$-partite** if the neighbourhood of every vertex induces an $r$-colourable graph.*

The Erdős-Simonovits problem for locally $r$-partite graphs is very natural: given a local colouring condition (the chromatic number of each neighbourhood), what can we say about the chromatic number of the whole graph?

Triangle-free graphs are exactly those in which every neighbourhood is an independent set (so is 1-colourable) and so these locally colourable graphs naturally extend the family of triangle-free graphs. On the other hand, all locally $r$-partite graphs are $K_{r+2}$-free so form an intermediate family between triangle-free and $K_{r+2}$-free graphs. A famous manifestation of this was Alon's [Alo96] result that any $n$-vertex locally $r$-partite graph with maximum degree $d$ has an independent set of size $\Omega_r(n \log d/d)$. This matched Ajtai, Komlós, and Szemerédi's [AKS80] result for triangle-free graphs (the best constant known is given by Shearer [She83]) and gave credence to their conjecture that it also held in the $K_{r+1}$-free case (where the best bound known of $\Omega_r(n \log d/(d \log \log d))$ is again due to Shearer [She95]).

Łuczak and Thomassé [ŁT10] gave an example to show that the chromatic threshold of locally bipartite graphs is at least $1/2$ and, furthermore, conjectured that there is equality. This conjecture was confirmed by Allen et al. [ABG+13]. Very little other progress has previously been made on the chromatic profile of locally colourable graphs. Indeed, the only other result is that of Jin, Liu, and Xu [JLX02] stating that any $n$-vertex locally bipartite graph with minimum degree greater than $7/12 \cdot n$ is 3-colourable (we improve this below). In this chapter, we make progress towards determining the chromatic profile of locally bipartite graphs, deferring the chromatic profile and threshold of more general locally colourable graphs to the next.

Locally bipartite graphs, just like triangle-free ones, exhibit a spectrum of thresholds. However, the structure of dense triangle-free and locally bipartite graphs do have some striking differences that we explore in § 3.1.2. In particular, these hint at added complications in the locally bipartite case.

Our understanding of their profile and structure can be summarised as follows. The graphs $H_2^+$, $H_2$, etc. can be seen in Figure 3.1 where they are discussed more thoroughly – for now, it suffices to say that they are all small, 4-chromatic, locally bipartite graphs.

**Theorem 3.2** (locally bipartite graphs)**.** *Let $G$ be a locally bipartite graph.*

- *If $\delta(G) > 4/7 \cdot |G|$, then $G$ is 3-colourable. Balanced blow-ups of $\overline{C}_7$ show that this is* *tight.*

- *If $\delta(G) > 5/9 \cdot |G|$, then $G$ is homomorphic to $\overline{C}_7$. Also, $G$ is either 3-colourable or contains $\overline{C}_7$. Suitable blow-ups of $H_2^+$ show that this is tight.*
- *There is an absolute constant $\varepsilon > 0$ such that if $\delta(G) > (5/9 - \varepsilon) \cdot |G|$, then $G$ is homomorphic to either $\overline{C}_7$ or $H_2^+$.*
- *If $\delta(G) > 6/11 \cdot |G|$, then $G$ is 4-colourable. Also, $G$ is either 3-colourable or contains $\overline{C}_7$ or $H_2^+$. In the first two cases, $G$ is homomorphic to $\overline{C}_7$.*
- *If $\delta(G) > 7/13 \cdot |G|$, then $G$ is either 3-colourable or contains $H_2$. Suitable blow-ups of $T_0$ show that this is tight.*
- *If $\delta(G) > 8/15 \cdot |G|$, then $G$ is either 3-colourable or contains $H_2$ or $T_0$. Suitable blow-ups of $H_1^{++}$ show that this is tight.*

This gives the following information about the chromatic profile of the family of locally bipartite graphs, which we denote by $\mathcal{F}_{1,2}$:

$$\delta_\chi(\mathcal{F}_{1,2}, 3) = 4/7, \quad \delta_\chi(\mathcal{F}_{1,2}, 4) \leqslant 6/11.$$

As mentioned above, Allen et al. [ABG+13], and Łuczak and Thomassé [ŁT10] combined to show that $\delta_\chi(\mathcal{F}_{1,2}) = 1/2$. Theorem 3.2 has extra structural results above and beyond the chromatic profile. The purpose of these is twofold. Firstly, the determination of the chromatic profile of triangle-free graphs was built upon many structural results and so these may aid future improvements to our understanding of locally bipartite graphs. Second and more importantly, they will be an ingredient in both determining the chromatic profile of locally colourable graphs in Chapter 4 and establishing the minimum degree stability of 4-chromatic $H$ in Chapter 5.

### 3.1.1 The Graphs

We owe the reader an introduction to the graphs mentioned in Theorem 3.2. They will appear frequently throughout this and the next two chapters. Here we note a few of their properties and show them in Figure 3.1

- All graphs shown are 4-chromatic and all bar $W_7$ are locally bipartite.
- The graph $H_0$ is isomorphic to the *Moser Spindle* – the smallest 4-chromatic unit distance graph. $H_0$ is also the smallest 4-chromatic locally bipartite graph and so it is natural that it should play such a pivotal role in many of our results. The graph $\overline{C}_7$ is the complement (and also the square) of the 7-cycle.
- Adding a single edge to $H_0$ while maintaining local bipartiteness can give rise to two non-isomorphic graphs, one of which is $H_1$. The other will appear fleetingly in Claim 3.15. Adding a single edge to $H_1$ while maintaining local bipartiteness

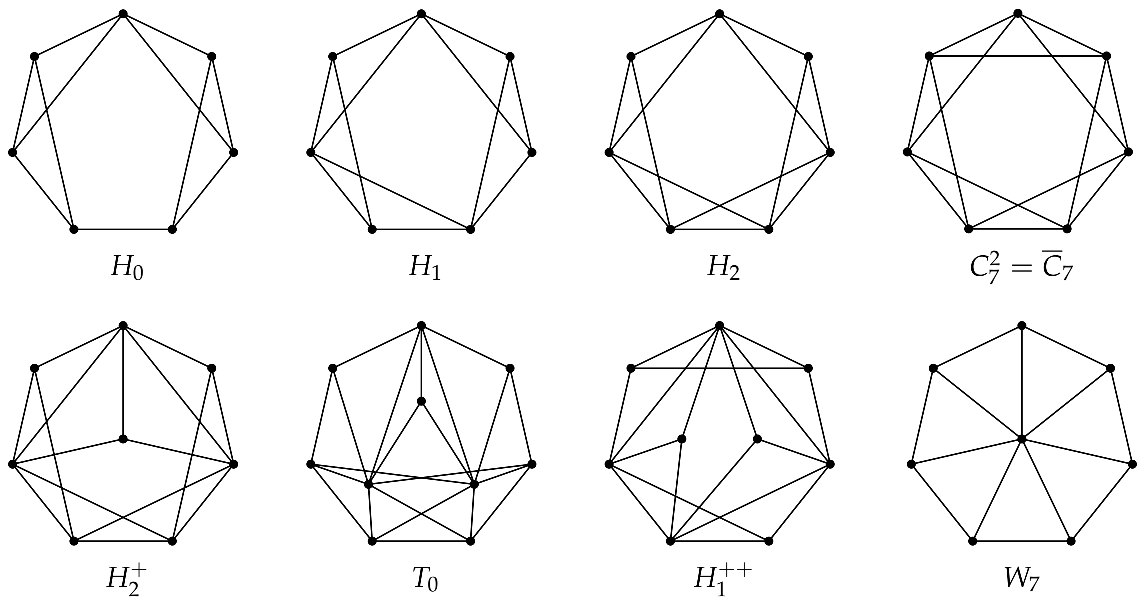
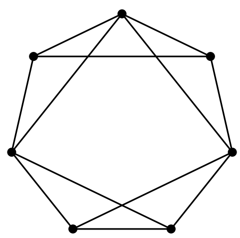



Figure 3.1

gives rise to a unique (up to isomorphism) graph – $H_2$. There is only one way to add a single edge to $H_2$ and maintain local bipartiteness – this gives $\overline{C}_7$. $H_2^+$ is $H_2$ with a degree 3 vertex added. $H_1^{++}$ is $H_1$ with two degree 3 vertices added.

- $\overline{C}_7$ and $H_2^+$ are both edge-maximal locally bipartite graphs.
- $T_0$ is a 7-cycle (the outer cycle) together with two vertices each joined to six of the seven vertices in the outer cycle (with the 'seventh' vertices two apart) and finally a vertex of degree three is added.
- $W_7$ is called the *7-wheel*. More generally, a single vertex joined to all the vertices of a $k$-cycle is called a *$k$-wheel* and is denoted by $W_k$. We term any edge from the central vertex to the cycle a *spoke* of the wheel and any edge of the cycle a *rim* of the wheel. Note that a graph is locally bipartite exactly if it does not contain an odd wheel (there is no such nice characterisation for a graph being locally tripartite, locally 4-partite, . . . ).

The graph $H_1$ has a symmetry that is not immediately apparent from Figure 3.1. The diagram below shows this symmetry more clearly and will be the usual way we display $H_1$.



The following observation gives a useful link between local bipartiteness and some of these graphs. We will use it frequently when copies of $H_0$, $H_1$, $H_2$, or $\overline{C}_7$ appear.

**Observation 3.3.** Any five vertices of $H_0$ contain a triangle or a 5-cycle. In particular, if $G$ is a locally bipartite graph, then any vertex can have at most four neighbours in any copy of $H_0$ appearing in $G$.

It will be useful to check whether one locally bipartite graph is homomorphic to another. The following lemma reduces 'homomorphic to' to 'contained in' when one of the graphs is edge-maximal and twin-free (recall this definition from § 1.3). Before proving this we should note that the family of locally bipartite graphs is both monotone and closed under taking blow-ups: if every neighbourhood in $G$ can be 2-coloured in $G$, then so can every neighbourhood in a blow-up of $G$. In particular, if $G$ is locally bipartite and $H \to G$, then $H$ is locally bipartite too (see the discussion at the end of § 1.3).

**Lemma 3.4.** *Let $F$ be a twin-free, edge-maximal locally bipartite graph. Let $F$ be homomorphic to a locally bipartite graph $G$. Then $F$ is an induced subgraph of $G$.*

*Proof.* Let $\varphi \colon F \to G$ be a homomorphism. If $\varphi$ is injective, then $F$ is a subgraph of $G$. But $G$ is locally bipartite and $F$ is edge-maximal locally bipartite, so any copy of $F$ appearing in $G$ must be induced. If $\varphi$ is not injective, then there are distinct vertices $u, v$ of $F$ with $\varphi(u) = \varphi(v)$. As $F$ is twin-free, we may assume there is a vertex $w$ in $F$ with $w$ adjacent to $v$ but not $u$. But then $\varphi$ is a homomorphism from $F + uw$ to $G$: $\varphi(u) = \varphi(v)$ and $\varphi(v)$ is adjacent to $\varphi(w)$.

However, $F$ is edge-maximal locally bipartite and $uw$ is not an edge of $F$, so $F + uw$ is not locally bipartite. But then $\varphi$ is a homomorphism from a graph that is not locally bipartite to a graph that is. $\square$

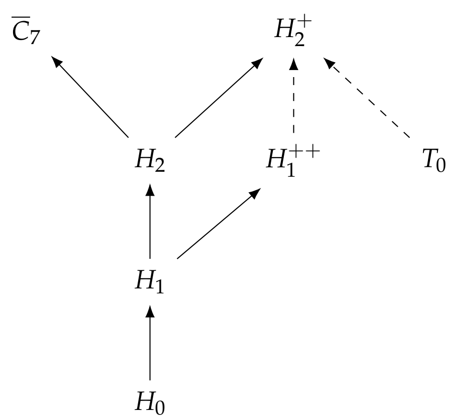
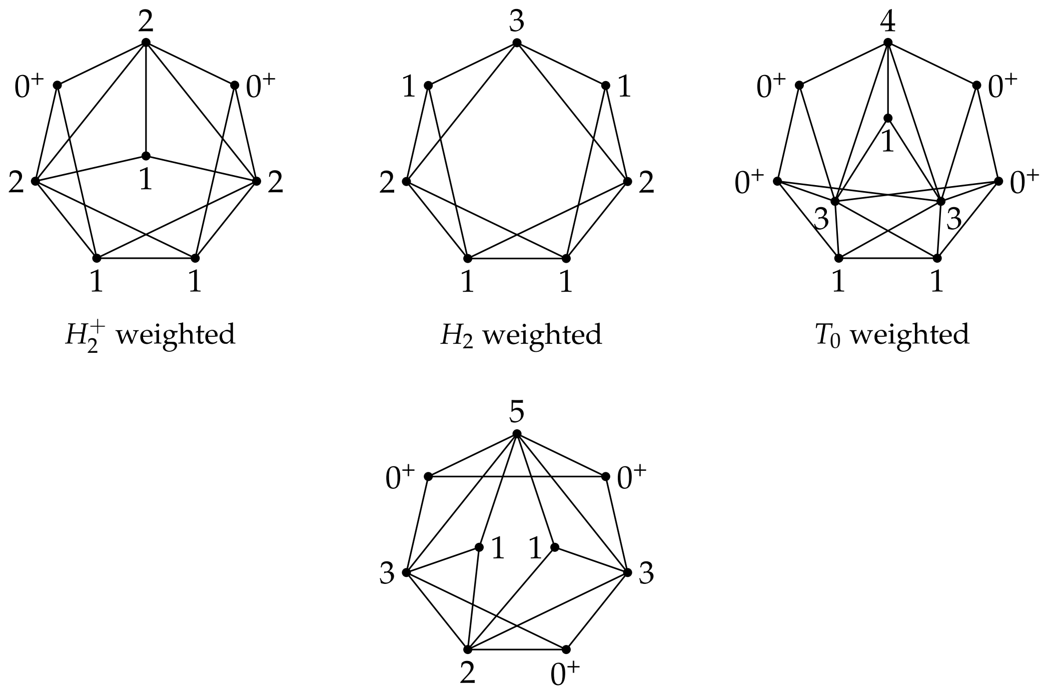



Figure 3.2

Containment and homomorphism relationships between the first seven graphs of Figure 3.1 are summarised in Figure 3.2. A full arrow pointing from $H$ to $G$ signifies that $H$ is a subgraph of $G$ and a dashed arrow from $H$ to $G$ signifies that $H$ is homomorphic to $G$. Furthermore, for two graphs $H$ and $G$ in the diagram, $H$ is homomorphic to $G$ exactly if there is a sequence of arrows starting at $H$ and ending at $G$. We verify Figure 3.2 in § 3.7.

Many of the graphs in Figure 3.1 are not regular while Theorem 3.2 makes reference to 'suitable blow-ups'. Below we give these blow-ups to which we will frequently refer. They are chosen so that the ratio between the minimum degree and order of the graph is as large as possible. When we 'weight a vertex by $0^+$' we are actually giving it a very small (but positive) rational weight – there is still a vertex there, we have not deleted it entirely.




$H_2^+$ weighted $\quad$ $H_2$ weighted $\quad$ $T_0$ weighted

$H_1^{++}$ weighted

Figure 3.3

As an example, the weighting of $H_2$ is a 6-regular graphs on 11 vertices, so there are $n$-vertex blow-ups of $H_2$ with minimum degree $\lfloor 6/11 \cdot n \rfloor$. Some care has to be taken for the $0^+$ weights. Consider the weighting of $H_2^+$ – at a glance it shows a 9-vertex graph with minimum degree 5, however, with the zero weights this is not strictly a blow-up of $H_2^+$. However it is the case that there are genuine $n$-vertex blow-ups of $H_2^+$ with minimum degree $5/9 \cdot n - \mathcal{O}(1)$. Similarly, there are $n$-vertex blow-ups of $T_0$ and $H_1^{++}$ with minimum degree $7/13 \cdot n - \mathcal{O}(1)$ and $8/15 \cdot n - \mathcal{O}(1)$, respectively.

From this we may obtain the tightness claims in Theorem 3.2. The complement of the

7-cycle is 4-regular and 4-chromatic so its balanced blow-ups do give the tightness of the first bullet point. For the second bullet point, there are $n$-vertex blow-ups of $H_2^+$ with minimum degree $5/9n - \mathcal{O}(1)$. These are 4-chromatic and locally bipartite (as $H_2^+$ is). They are also $\overline{C}_7$-free, as $\overline{C}_7$ is not homomorphic to $H_2^+$. Similar reasoning applies for the fifth and sixth bullet points.

### 3.1.2 Comparison with the Triangle-free Case and some Open Questions

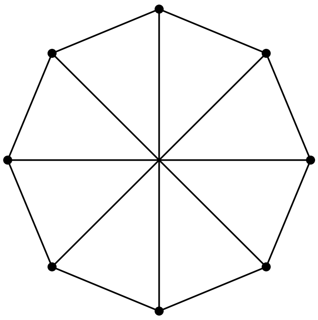
Here we explain the structural results behind the complete determination of the chromatic profile of triangle-free graphs and compare them to those in Theorem 3.2.

The proper way to discuss the chromatic profile of triangle-free graphs (and indeed the route taken by Brandt and Thomassé [BT05]) is through weightings. We refer the reader to § 1.3 for definitions. The key observation is that if a weighted graph has a pair of twins, then merging those vertices and giving the new vertex the sum of their weights produces an equivalent graph with the same total weight.

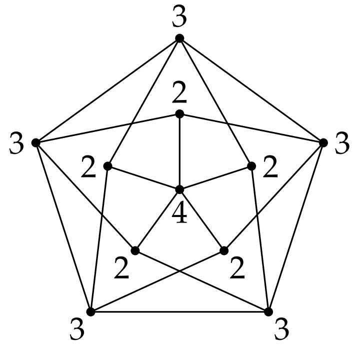
Suppose we start with a triangle-free graph $G$. We can repeatedly add edges to $G$ and merge twins to obtain a twin-free, edge-maximal triangle-free weighted graph $(H, \omega)$ whose total weight, $\omega(H)$, equals $|G|$ and whose minimum degree, $\delta(H, \omega)$, is at least $\delta(G)$. Note that $G$ is homomorphic to $H$. In particular, to understand the chromatic profile of triangle-free graphs, one only needs to understand the twin-free, edge-maximal triangle-free graphs $H$ that have a weighting $\omega$ with $\delta(H, \omega) > 1/3 \cdot \omega(H)$ (we will refer to this last property as '$H$ beating $1/3$'). The above reasoning holds if we replace 'triangle-free' by 'locally bipartite' and replace $1/3$ by $1/2$ (the corresponding chromatic threshold). Hence, we are particularly interested in twin-free, edge-maximal locally bipartite graphs that beat $1/2$. Lemma 3.4 applies and so for two such graphs $G$ and $H$, $G$ is homomorphic $H$ if and only if $G$ is an induced subgraph of $H$.

In the triangle-free case, the endeavour of finding all such graphs was implicitly started by Häggkvist [Häg82]. He produced a suitable weighting of the Grötzsch graph to disprove Erdős and Simonovits's original conjecture and proved that all triangle-free graphs beating $3/8$ are homomorphic to the 5-cycle. He noted that $3/8$ is tight due to the Möbius ladder on eight vertices, $M_8$, which consists of an 8-cycle together with chords joining opposite vertices of the cycle.

He further conjectured that every triangle-free graph beating $10/29$ (the number corresponding to his weighting of the Grötzsch graph) is homomorphic to $M_8$. This last conjecture turned out to be false but his focus on homomorphisms was a key insight for taming the chromatic number and structure of triangle-free graphs beating 1/3.



$M_8$




Grötzsch graph with Häggkvist's weighting

Figure 3.4

The 5-cycle and Möbius ladder are graphs in an infinite sequence of triangle-free graphs introduced much earlier by Andrásfai [And62]. These *Andrásfai graphs* $\Gamma_1 = K_2, \Gamma_2 = C_5, \Gamma_3 = M_8, \ldots$ are all twin-free, edge-maximal triangle-free graphs beating 1/3 (and $\Gamma_4$ provides a counterexample to Häggkvist's conjecture). The graph $\Gamma_i$ can be viewed in a couple of ways, both starting with $3i-1$ vertices $a_1, a_2, \ldots, a_{3i-1}$ equally spaced around a circle (we consider indices modulo $3i-1$). In the first way, $a_x$ is joined to $a_{x+i}$, $a_{x+i+1}$, $\ldots, a_{x+2i-1}$ (this shows that $\Gamma_i$ is the complement of $(i-1)^{\text{th}}$ power of the $(3i-1)$-cycle, $C_{3i-1}^{i-1}$). In the second way, $a_x$ is joined to $a_{x+1}, a_{x+4}, a_{x+7}, \ldots, a_{x-4}, a_{x-1}$. It is $i$-regular on $3i-1$ vertices and all bar the first have chromatic number three (it has independence number $i$).

The next great advance was by Jin [Jin93] who showed that all triangle-free graphs beating 10/29 are homomorphic to one of the first nine Andrásfai graphs (and hence are 3-colourable). Together with Chen and Koh [CJK97], he then characterised all 3-colourable, twin-free, edge-maximal triangle-free graphs beating 1/3 – they are exactly the Andrásfai graphs. At that point only two non-3-colourable, twin-free, edge-maximal triangle-free graphs beating 1/3 were known: the Grötzsch graph and one found by Jin [Jin95]. Then, using the computer programme *Vega*, Brandt and Pisanski [BP98] found an infinite sequence of such graphs all of which are 4-chromatic. These *Vega graphs*, which we denote by $\Upsilon_j$, are essentially obtained by taking an Andrásfai graph and tacking on eight vertices in a clever way to increase the chromatic number while not sacrificing the minimum degree too much. Next, Brandt [Bra02] showed that all regular triangle-free graphs beating 1/3 are 4-colourable and finally Brandt and Thomassé [BT05] showed that the twin-free, edge-maximal triangle-free graphs beating 1/3 are exactly the Andrásfai and Vega graphs. Hence every triangle-free $G$ with $\delta(G) > 1/3 \cdot |G|$ is homomorphic to one of these (and so is 4-colourable). Now for any $c > 1/3$, only finitely many graphs of each sequence beat $c$. In particular, if a triangle-free $G$ has $\delta(G)/|G| \geqslant c$ for $c > 1/3$, then $G$ is homomorphic to some early $\Gamma_i$ or to some early $\Upsilon_j$.

Theorem 3.2 effectively shows that $K_3$, $\overline{C}_7$, $H_2^+$ play the same role for locally bipartite graphs as the first three Andrásfai graphs do for triangle-free graphs. Furthermore, the theorem together with Lemma 3.4 show that they are the only twin-free, edge-maximal locally bipartite graphs which beat $5/9 - \varepsilon$ (in fact, we believe this is true down to $6/11$). These results display similarities with the triangle-free case but also give a couple of striking differences. Firstly, the Andrásfai graphs are nested, while $H_2^+$ does not contain $\overline{C}_7$ and so, by Lemma 3.4, neither is even homomorphic to the other. Secondly, the Andrásfai and Vega graphs have weightings in which all vertices have the same degree as is expected for extremal examples. However, $H_2^+$ has no such weighting and, in fact, its $n$-vertex weighting with greatest minimum degree $(5/9 \cdot n)$ has $1/9 \cdot n$ vertices with degree $2/3 \cdot n$ (this was displayed in Figure 3.3 on page 33).

It is natural to ask what graphs come after $H_2^+$. There is an infinite nested sequence of twin-free, edge-maximal locally bipartite graphs all beating $1/2$: define $\Delta_\ell$ as the complement of $C_{4\ell-1}^{\ell-1}$ (this is analogous to $\Gamma_i$, which is the complement of $C_{3i-1}^{i-1}$). Then $\Delta_\ell$ has $4\ell - 1$ vertices, is $(2\ell)$-regular, is 4-chromatic (its independence number is $\ell$), and is edge-maximal locally bipartite (the addition of any edge gives a 4-clique). Note that $\Delta_2 = \overline{C}_7$. In fact, $\Delta_3$ satisfies $\delta(\Delta_3)/|\Delta_3| = 6/11$ suggesting it is the next key graph when extending the colourability results of Theorem 3.2 below $6/11$. Unlike the triangle-free case, the $\Delta_\ell$ are not the only 4-chromatic, twin-free, edge-maximal locally bipartite graphs beating $1/2$. Indeed, $H_2^+$ is not a $\Delta_\ell$ and nor is the graph shown in Figure 3.6 on page 69. Intriguingly, neither of these graphs is contained in (nor, by Lemma 3.4, homomorphic to) any $\Delta_\ell$, since no $\Delta_\ell$ contains an induced $H_2$ (no neighbourhood in $\Delta_\ell$ contains two edges with no edges between). It would be interesting to have an infinite sequence of such non-$\Delta_\ell$ graphs. Also, for each $c > 1/2$, are there only finitely many twin-free, edge-maximal locally bipartite graphs beating $c$ (for triangle-free graphs this was first shown by Łuczak [Łuc06])?

A final question is whether there are any locally bipartite graphs corresponding to the Vega graphs or if, in fact, $\delta_\chi(\mathcal{F}_{1,2}, 5) = \delta_\chi(\mathcal{F}_{1,2}) = 1/2$.

### 3.1.3 Notation

Let $G$ be a graph and $X \subset V(G)$ a set of vertices. We write $e(X, G)$ for the number of ordered pairs of vertices $(x, v)$ with $x \in X$, $v \in G$ and $xv$ an edge of $G$. In particular, $e(X, G)$ counts each edge in $G[X]$ twice and each edge from $X$ to $V(G) \setminus X$ once. It satisfies

$$e(X, G) = \sum_{x \in X} d(x) = \sum_{v \in G} |\Gamma(v) \cap X|.$$

We generalise this notation to vertex weightings which are used in many of our arguments. We will assign weights $\omega\colon X \to \mathbb{Z}^+$ to the vertices of $X$. Then we define

$$\omega(X, G) = \sum_{x \in X} \omega(x) d(x) = \sum_{v \in G} \text{Total weight of the neighbours of } v \text{ in } X.$$

We will often use the word *circuit* (as opposed to cycle) in this chapter's arguments. A circuit is a sequence of (not necessarily distinct) vertices $v_1, v_2, \ldots, v_\ell$ with $\ell \geqslant 3$, $v_i$ adjacent to $v_{i+1}$ (for $i = 1, 2, \ldots, \ell - 1$) and $v_\ell$ adjacent to $v_1$. Note that in a locally bipartite graph the neighbourhood of any vertex does not contain an odd circuit (and of course does not contain an odd cycle). We use circuit to avoid considering whether some pairs of vertices are distinct when it is unnecessary to do so.

## 3.2 Initial Observations and a Sketch of $\delta_\chi(\mathcal{F}_{1,2}, 3) = 4/7$

We will now motivate the proof of Theorem 3.2 by making some initial observations, definitions as well as giving a sketch proof of the first bullet point – that is, of $\delta_\chi(\mathcal{F}_{1,2}, 3) = 4/7$. This result corresponds to the simple fact that $\delta_\chi(K_3, 2) = 2/5$. Although that fact has a very short proof indeed (see Lemma 5.9), a more substantial argument is required here.

We start with a locally bipartite graph $F$ with $\delta(F) > 4/7 \cdot |F|$ and wish to show that $F$ is 3-colourable. We may as well assume that $F$ is edge-maximal. That is, the addition of any edge will create a vertex with a non-bipartite neighbourhood. Thus, any non-edge of $F$ is either a missing edge of a $K_4$, a missing rim of an odd wheel, or a missing spoke of an odd wheel. This motivates a key definition.

**Definition 3.5** (dense and sparse)**.** *A pair of* non-adjacent, distinct *vertices $u$, $v$ in a graph $G$ is* ***dense*** *if $G_{u,v}$ contains an edge and* ***sparse*** *if $G_{u,v}$ does not contain an edge.*

First note that every pair of distinct vertices in any graph is exactly one of 'adjacent', 'dense' or 'sparse'. Another way to view being dense is as being the missing edge of a $K_4$. Locally bipartite graphs are $K_4$-free so any pair of distinct vertices with an edge in their common neighbourhood must be non-adjacent and so must be dense. In particular, in locally bipartite graphs, to establish that a pair is dense does not require checking that they are non-adjacent. Our initial observations above show that, in an edge-maximal locally bipartite graph, each pair of sparse vertices is either the missing rim or spoke of an odd wheel. The distinction between dense and sparse pairs turns out to be crucial.

We collect three simple but very effective lemmas about dense and sparse pairs of vertices. The second of these exhibits a very simple edge counting method that we will use frequently. The final lemma hints at the importance of $H_0$ in our arguments.

**Lemma 3.6.** *Let $G$ be a graph with $\delta(G) > 1/2 \cdot |G|$ and let $I$ be any largest independent set in $G$. Then, for every distinct $u, v \in I$, the pair $u, v$ is dense.*

*Proof.* Fix distinct $u, v \in I$ – first note that these are not adjacent. Now $\Gamma(u), \Gamma(v) \subset V(G) \setminus I$ so $|\Gamma(u) \cup \Gamma(v)| \leqslant |G| - |I|$. Hence

$$\begin{aligned}|\Gamma(u) \cap \Gamma(v)| &= d(u) + d(v) - |\Gamma(u) \cup \Gamma(v)| \geqslant 2\delta(G) - (|G| - |I|)\\ &= |I| + 2\delta(G) - |G| > |I|.\end{aligned}$$

But $I$ is a largest independent set in $G$ and so $\Gamma(u) \cap \Gamma(v)$ is not independent: $G_{u,v}$ contains an edge so the pair $u, v$ is dense. □

**Lemma 3.7.** *Let $G$ be a graph with $\delta(G) > 1/2 \cdot |G|$ and suppose $C$ is an induced 4-cycle in $G$. Then at least one of the non-edges of $C$ is a dense pair.*

*Proof.* Suppose the result does not hold. We have an induced 4-cycle $C = v_1v_2v_3v_4$ in $G$ with edges $v_1v_2$, $v_2v_3$, $v_3v_4$, $v_4v_1$ where the pairs $v_1, v_3$ and $v_2, v_4$ are both sparse. Note that any vertex has at most two neighbours in $C$. Indeed if $u$ is adjacent to both $v_1$ and $v_3$, then $u$ cannot be adjacent to either $v_2$ or $v_4$, as the pair $v_1, v_3$ is sparse; similarly, if $u$ is adjacent to both $v_2$ and $v_4$, then $u$ cannot be adjacent to either $v_1$ or $v_3$. Counting the edges between $C$ and $G$ from both sides gives

$$4\delta(G) \leqslant d(v_1) + d(v_2) + d(v_3) + d(v_4) = e(C, G) \leqslant 2 \cdot |G|,$$

which contradicts $\delta(G) > 1/2 \cdot |G|$. □

**Lemma 3.8.** *Let $G$ be a locally bipartite graph that does not contain $H_0$. For any vertex $v$ of $G$,*

$$D_v := \{u \colon \textit{the pair } u, v \textit{ is dense}\}$$

*is an independent set of vertices.*

*Proof.* Suppose that in fact there are distinct vertices $v$, $u_1$ and $u_2$ with the pairs $v, u_1$ and $v, u_2$ both dense and with $u_1$ adjacent to $u_2$. Let $x_1x_2$ be an edge in the common neighbourhood of $v$ and $u_1$ and $x_3x_4$ be an edge in the common neighbourhood of $v$ and $u_2$.

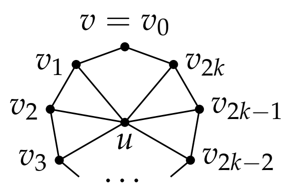
If $\{x_1, x_2\} = \{x_3, x_4\}$, then $u_1u_2x_1x_2$ is a $K_4$ in $G$. If $\{x_1, x_2\}$ and $\{x_3, x_4\}$ have one element in common, say $x_1 = x_3$, then $G_{x_1}$ contains the 5-cycle $vx_2u_1u_2x_4$. Hence $\{x_1, x_2\}$ and $\{x_3, x_4\}$ must be disjoint. But then $G[\{v, x_1, x_2, u_1, u_2, x_3, x_4\}]$ contains a copy of $H_0$. $\square$

We continue proving $\delta_\chi(\mathcal{F}_{1,2}, 3) = 4/7$. The graph $F$ is edge-maximal locally bipartite with $\delta(F) > 4/7 \cdot |F|$. We first show that $F$ does not contain $H_0$. Suppose that a copy of $H_0$ appeared in $F$ with vertex set $X$ (so $|X| = 7$). By Observation 3.3, every vertex of $F$ has at most four neighbours in $X$. Thus

$$7\delta(F) \leqslant e(X, F) \leqslant 4|F|,$$

which is a contradiction. Hence $F$ is $H_0$-free.

By edge-maximality, all sparse pairs of $F$ are the missing rim or spoke of an odd wheel. We now rule out any sparse pair being the missing spoke of an odd wheel. Indeed, suppose that $uv$ is the missing spoke of an odd wheel with $u, v$ sparse. We may assume that the length of the wheel is minimal (subject to having centre $u$ and passing through $v$).




Let $C = \{v, v_1, \ldots, v_{2k}\}$ denote the outer $(2k+1)$-cycle. By minimality of $k$ and the sparsity of the pair $u, v$, any neighbour of $u$ has at most two neighbours in $C$ (see Lemma 3.21 for more details). Also any vertex has at most $2k$ neighbours in $C$, as $F$ does not contain an odd wheel. Thus

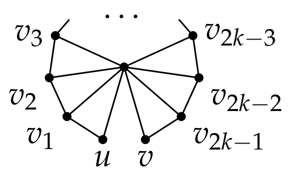
$$\begin{aligned}(2k+1)\delta(F) &\leqslant e(C, F) \leqslant 2d(u) + 2k(|F| - d(u)) \\ &= 2k|F| - (2k-2)d(u) \leqslant 2k|F| - (2k-2)\delta(F),\end{aligned}$$

so $4/7 < \delta(F)|F|^{-1} \leqslant 2k/(4k-1)$ which implies $k < 2$. Thus any missing edge is either dense or is the missing rim of an odd wheel.

Let $I$ be a largest independent set in $F$ and fix any $u \in I$. By Lemma 3.6, $I \subset D_u \cup \{u\}$. On the other hand, $F$ is $H_0$-free, and so, by Lemma 3.8 and the definition of density,

$D_u \cup \{u\}$ is an independent set. By the maximality of $I$, we must have $I = D_u \cup \{u\}$. This holds for all $u \in I$. In particular, if the pair $u, v$ is dense and $u \in I$, then $v \in I$ also.

Our final claim is that every vertex of $F$ is either in $I$ or adjacent to all of $I$. The 3-colourability of $F$ will follow immediately: fix some $u \in I$ and note that $F[V(F) \setminus I] = F_u$ must be bipartite and so giving a third colour to $I$ produces a valid 3-colouring of $F$. To prove the claim, fix $u \in I$ and let $v$ be a vertex which is not adjacent to $u$. If the pair $u, v$ is dense, then $v \in I$, as required. If the pair $u, v$ is sparse, then $uv$ is the missing rim of some odd wheel.




The pair $u, v_2$ is dense (if they are adjacent, then a $K_4$ is present) and $u \in I$, so $v_2 \in I$ as well. The pair $v_2, v_4$ is dense and so $v_4 \in I$ as well. Repeating this gives $v_r \in I$ for all even $r$. In particular, $v \in I$, as required. $\square$

## 3.3 The Components of Theorem 3.2

At the end of § 3.1.1 we proved all the tightness claims of Theorem 3.2. The rest of our understanding of locally bipartite graphs can be summarised in the following five results: Theorems 3.9 to 3.13. We will show shortly that Theorem 3.2 follows from these.

**Theorem 3.9.** *Let $G$ be a locally bipartite graph. If $\delta(G) > 8/15 \cdot |G|$, then $G$ is either 3-colourable or contains $H_0$ or $T_0$. If $\delta(G) > 7/13 \cdot |G|$, then $G$ is either 3-colourable or contains $H_0$.*

We note that the blow-up of $T_0$ in Figure 3.3 on page 33 shows that $7/13$ is tight (recall from § 3.1.1 that $H_0$ is not homomorphic to $T_0$, so this blow-up is $H_0$-free).

**Theorem 3.10.** *Let $G$ be a locally bipartite graph that contains $H_0$.*

- *Firstly, it must be the case that $\delta(G) \leqslant 4/7 \cdot |G|$.*
- *Secondly, if $\delta(G) > 5/9 \cdot |G|$, then $G$ contains $\overline{C}_7$.*
- *Thirdly, if $\delta(G) > 6/11 \cdot |G|$, then $G$ contains $H_2^+$ or $\overline{C}_7$.*
- *Fourthly, if $\delta(G) > 8/15 \cdot |G|$, then $G$ contains $H_2$.*
- *Finally, if $\delta(G) > 1/2 \cdot |G|$, then $G$ contains $H_1$.*

The graph $\overline{C}_7$ (and any of its balanced blow-ups) show that $4/7$ is tight. The blow-ups of $H_2^+$, $H_2$, and $H_1^{++}$ in Figure 3.3 on page 33 show respectively that $5/9$, $6/11$, and $8/15$ are tight (recall from § 3.1.1 that $H_2^+$ is not homomorphic to $\overline{C}_7$; $H_2^+$ and $\overline{C}_7$ are both not homomorphic to $H_2$; and $H_2$ is not homomorphic to $H_1^{++}$).

**Theorem 3.11.** *Let $G$ be a locally bipartite graph. If $\delta(G) > 6/11 \cdot |G|$ and $G$ contains $\overline{C}_7$, then $G$ is homomorphic to $\overline{C}_7$.*

**Theorem 3.12.** *Let $G$ be a locally bipartite graph. If $\delta(G) > 6/11 \cdot |G|$, then $G$ is 4-colourable.*

**Theorem 3.13.** *There is an $\varepsilon > 0$ such that if $G$ is a locally bipartite graph with $\delta(G) > (5/9 - \varepsilon) \cdot |G|$ and $G$ does not contain $\overline{C}_7$, then $G$ is homomorphic to $H_2^+$.*

**Remark 3.14.** We make no attempt to optimise the proof to obtain the 'best value' of $\varepsilon$ as we believe that it is in fact possible to replace $5/9 - \varepsilon$ by $6/11$ in Theorem 3.13.

*Proof of Theorem 3.2.* As noted at the start of this section, we already have the tightness claims. Let $G$ be a locally bipartite graph. Theorem 3.9 and Theorem 3.10 together show that

- If $\delta(G) > 4/7 \cdot |G|$, then $G$ is 3-colourable.
- If $\delta(G) > 5/9 \cdot |G|$, then $G$ is either 3-colourable or contains $\overline{C}_7$.
- If $\delta(G) > 6/11 \cdot |G|$, then $G$ is either 3-colourable, contains $H_2^+$, or contains $\overline{C}_7$.
- If $\delta(G) > 7/13 \cdot |G|$, then $G$ is either 3-colourable or contains $H_2$.
- If $\delta(G) > 8/15 \cdot |G|$, then $G$ is either 3-colourable, contains $H_2$, or contains $T_0$.

This immediately gives everything in Theorem 3.2 except the homomorphism statements. Thus we may assume that $\delta(G) > 6/11 \cdot |G|$. If $G$ is 3-colourable, then it is certainly homomorphic to $\overline{C}_7$ (as this contains a triangle). Also if $G$ contains $\overline{C}_7$, then, by Theorem 3.11, $G$ is homomorphic to $\overline{C}_7$.

We are left with the case where $G$ is $\overline{C}_7$-free and not 3-colourable. The second bullet point above shows that $\delta(G) \leqslant 5/9 \cdot |G|$ and so the second bullet point of Theorem 3.2 holds. Theorem 3.13 gives the third and Theorem 3.12 the fourth. □

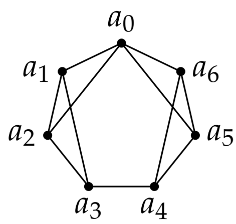
In § 3.4 we carry out a careful edge counting/vertex weighting argument which proves Theorem 3.10. In § 3.5 we rule out some configurations from appearing in $G$ and in § 3.6 we prove Theorem 3.9. The proofs of Theorems 3.11 to 3.13 are deferred to § 3.8.

## 3.4 From $H_0$ to $\overline{C}_7$

In this section, we prove Theorem 3.10. For ease of reading we split the proof into a sequence of claims from $1/2$ up to $4/7$. Each claim corresponds to a bullet point of Theorem 3.10 and we are addressing the bullet points in reverse order. The proofs of the first two claims are the longest.

**Claim 3.15.** *Let $G$ be a locally bipartite graph containing $H_0$. If $\delta(G) > 1/2 \cdot |G|$, then $G$ contains $H_1$.*

*Proof.* We label a copy of $H_0$ in $G$ as below and let $X = \{a_0, a_1, \ldots, a_6\}$.




Let $U_4$ be the set of vertices with exactly four neighbours in $X$. Observation 3.3 shows that no vertex has five neighbours in a copy of $H_0$, so every non-$U_4$ vertex has at most three neighbours in $X$. Hence

$$7/2 \cdot |G| < 7\delta(G) \leqslant e(X, G) \leqslant 4|U_4| + 3(|G| - |U_4|) = 3|G| + |U_4|,$$

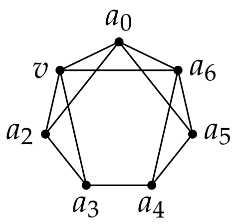
and so

$$|U_4| > 1/2 \cdot |G|.$$

Now $|U_4| + d(a_0) > |G|$ and so some vertex $v$ is adjacent to $a_0$ and has four neighbours in $X$. Note that $v$ cannot be adjacent to both $a_1$, $a_2$ as otherwise $va_0a_1a_2$ is a $K_4$, so by symmetry we may assume that $v$ is not adjacent to $a_1$. Similarly we may assume that $v$ is not adjacent to $a_5$. But $v$ has four neighbours in $X$ so must be adjacent to at least one of $a_2$, $a_6$ – by symmetry we may assume $v$ is adjacent to $a_2$.

There are two possibilities: $v$ is adjacent to $a_0$, $a_2$, $a_3$, $a_4$, or $v$ is adjacent to $a_0$, $a_2$, $a_6$ and one of $a_3$, $a_4$. Suppose the former case occurs. If $v = a_5$, then $a_3a_5$ is an edge and so $G[X]$ contains $H_1$. Similarly if $v = a_6$. If $v$ is neither $a_5$ nor $a_6$, then $G[X \setminus \{a_1\} \cup \{v\}]$ contains $H_1$.

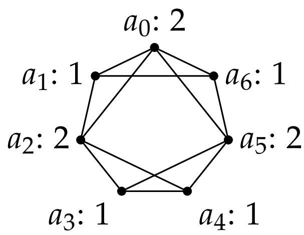
Now suppose the latter case occurs. We may assume by symmetry that $v$ is adjacent to $a_3$. If $v = a_4$, then $a_0a_4$ is an edge and so $G$ contains a $K_4$. If $v = a_5$, then $a_3a_5$ is an edge and so $G[X]$ contains $H_1$. If $v$ is neither $a_4$ nor $a_5$, then $G$ contains the following graph.




If $v$ is adjacent to $a_4$, then $G$ contains a copy of $H_1$ (ignore the edge $va_6$). Similarly if $a_3$ is adjacent to $a_6$. Otherwise $va_6a_4a_3$ is an induced 4-cycle in $G$. By Lemma 3.7, at least one of the pairs $v, a_4$ and $a_3, a_6$ is dense. By symmetry we may assume that $v, a_4$ is dense: let $a_2'a_3'$ be an edge in $G_{v,a_4}$. Note that $a_5$ is not adjacent to $v$ (else $a_5a_6a_0v$ is a $K_4$) so $a_5$ is neither $a_2'$ nor $a_3'$. Similarly $a_0$ is not adjacent to $a_4$ and so $a_0$ is neither $a_2'$ nor $a_3'$. If $a_6 = a_2'$, then $G_{a_6}$ contains the 5-cycle $a_3'va_0a_5a_4$, which is impossible. Hence, $a_2'$, $a_3'$ are distinct from $a_4$, $a_5$, $a_6$, $a_0$, $v$ and so $G[\{a_6, a_0, v, a_2', a_3', a_4, a_5\}]$ contains a copy of $H_1$ (with apex $a_6$). Hence in all cases $G$ contains a copy of $H_1$. $\square$

**Claim 3.16.** *Let $G$ be a locally bipartite graph containing $H_1$. If $\delta(G) > 8/15 \cdot |G|$, then $G$ contains $H_2$.*

*Proof.* Consider a copy of $H_1$ with vertices $X = \{a_0, a_1, \ldots, a_6\}$. We assign weights $\omega\colon X \to \mathbb{Z}^+$ as shown in the diagram below, so, for example $\omega(a_0) = 2$ and $\omega(a_1) = 1$ (recall this notation from § 3.1.3). We will often use diagrams to give weightings in this way. For each vertex $v \in G$, let $f(v)$ be the total weight of the neighbours of $v$ in $X$.




We will assume that $G$ does not contain $H_2$. We first show that every vertex $v$ has $f(v) \leqslant 6$ and further that if $f(v) = 6$, then $\Gamma(v) \cap X = \{a_1, a_2, a_5, a_6\}$.

Let $v$ be a vertex with $f(v) \geqslant 6$. No vertex has five neighbours in a copy of $H_1$ (as noted by Observation 3.3), so $v$ is adjacent to at most four of the $a_i$. Thus, $v$ is adjacent to at least two of the vertices of weight two, that is, to at least two of $a_0$, $a_2$, $a_5$. If $v$ is adjacent to all of $a_0$, $a_2$, $a_5$, then $G_{a_0}$ contains the odd circuit $va_5a_6a_1a_2$. Thus $v$ is adjacent to exactly two of $a_0$, $a_2$, and $a_5$.

Suppose $v$ is adjacent to $a_0$. By symmetry we may assume that $v$ is adjacent to $a_2$ but not to $a_5$. Then $v$ is not adjacent to $a_1$, else $va_0a_1a_2$ is a $K_4$. Similarly $v$ cannot be adjacent to both $a_3$ and $a_4$. Hence $v$ is adjacent to $a_0$, $a_2$, $a_6$ and one of $a_3$, $a_4$. By symmetry, we may assume $v$ is adjacent to $a_3$. Now $v$ cannot be $a_4$ nor $a_5$ as it would then have five neighbours in $X$. Replacing $a_1$ by $v$ gives a copy of $H_2$ in $G$.

Thus $v$ is not adjacent to $a_0$ and so is adjacent to both $a_2$ and $a_5$. Note $v$ cannot be adjacent to both $a_3$, $a_4$ else $va_2a_3a_4$ is $K_4$, so we may assume that $v$ is adjacent to $a_1$. If $v$ were adjacent to one of $a_3$, $a_4$, then we may assume, by symmetry that $v$ is adjacent to $a_4$ and not $a_3$. Now $v$ cannot be $a_0$ nor $a_6$ as it would then have five neighbours in $X$. Hence $G$ contains $H_2$ (replace $a_3$ by $v$). Therefore $v$ is adjacent to neither $a_3$ nor $a_4$. But $f(v) \geqslant 6$, so $v$ is adjacent to $a_1$, $a_6$ and thus $\Gamma(v) \cap X = \{a_1, a_2, a_5, a_6\}$.

So every vertex $v$ has $f(v) \leqslant 6$ and furthermore all $v \in \Gamma(a_0) \cup \Gamma(a_3) \cup \Gamma(a_4)$ have $f(v) \leqslant 5$. We first claim that there is $i \in \{3, 4\}$ such that $\Gamma(a_0, a_2, a_i) = \varnothing$. If not, there is $a_1' \in \Gamma(a_0, a_2, a_3)$ and $a_1'' \in \Gamma(a_0, a_2, a_4)$. But then $G_{a_2}$ contains the odd circuit $a_1'a_3a_4a_1''a_0$. Similarly there is $j \in \{3, 4\}$ such that $\Gamma(a_0, a_5, a_j) = \varnothing$.

Next we claim that there is $i \in \{3, 4\}$ with $\Gamma(a_0, a_2, a_i) = \Gamma(a_0, a_5, a_i) = \varnothing$. If not, then without loss of generality there is $a_1' \in \Gamma(a_0, a_2, a_3)$ and $a_6' \in \Gamma(a_0, a_5, a_4)$. Certainly, $\Gamma(a_1') \cap X \neq \{a_1, a_2, a_5, a_6\}$, so $f(a_1') \leqslant 5$. But $\omega(a_0) + \omega(a_2) + \omega(a_3) = 5$, so $\Gamma(a_1') \cap X = \{a_0, a_2, a_3\}$. Similarly $\Gamma(a_6') \cap X = \{a_0, a_4, a_5\}$. In particular, all of $a_0, \ldots, a_6$, $a_1'$, $a_6'$ are distinct. But then $G[\{a_0, a_1', a_2, a_3, a_4, a_5, a_6'\}]$ contains $H_2$. Thus, without loss of generality, $\Gamma(a_0, a_2, a_3) = \Gamma(a_0, a_5, a_3) = \varnothing$. Then $\Gamma(a_3)$, $\Gamma(a_0, a_2)$, $\Gamma(a_0, a_5)$ are pairwise disjoint (we already showed that $\Gamma(a_0, a_2, a_5) = \varnothing$). Hence the set $Y = \Gamma(a_3) \cup \Gamma(a_0, a_2) \cup \Gamma(a_0, a_5)$ has size

$$|Y| = d(a_3) + d(a_0, a_2) + d(a_0, a_5) \geqslant \delta(G) + 2(2\delta(G) - |G|) = 5\delta(G) - 2|G|.$$

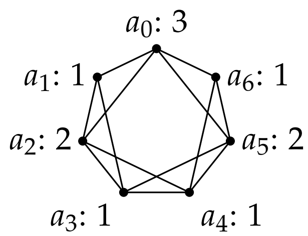
Now, all $v \in Y$ have $f(v) \leqslant 5$. We bound $\omega(X, G)$ from both directions (recalling this notation from § 3.1.3) to get

$$10\delta(G) \leqslant \sum_{x \in X} \omega(x) d(x) = \sum_{v \in G} f(v) \leqslant 5|Y| + 6(|G| - |Y|) = 6|G| - |Y| \leqslant 8|G| - 5\delta(G),$$

which contradicts $\delta(G) > 8/15 \cdot |G|$. □

**Claim 3.17.** *Let $G$ be a locally bipartite graph containing $H_2$. If $\delta(G) > 6/11 \cdot |G|$, then $G$ contains either $H_2^+$ or $\overline{C}_7$.*

*Proof.* Consider a copy of $H_2$ in $G$ with vertices $X = \{a_0, a_1, \ldots, a_6\}$. Assign weights $\omega(a_i)$ to the vertices of $X$ as shown in the diagram. For each vertex $v \in G$, let $f(v)$ be the total weight of the neighbours of $v$ in $X$.




Now

$$6|G| < 11\delta(G) \leqslant \sum_{v \in G} f(v),$$

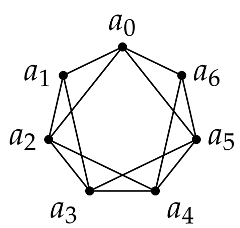
so some vertex $v$ has $f(v) \geqslant 7$. But all vertices have at most four neighbours in a copy of $H_2$, so either $v$ is adjacent to all of $a_0$, $a_2$, $a_5$ or $v$ is adjacent to $a_0$, to exactly one of $a_2$ and $a_5$, and to exactly two of $a_1$, $a_3$, $a_4$, $a_6$.

First suppose that $v$ is adjacent to all of $a_0$, $a_2$, and $a_5$. Note that $v$ cannot be in $X$. Indeed, if $v = a_1$, then $G_{a_5}$ contains the 5-cycle $a_0va_3a_4a_6$ and similarly if $v = a_6$. On the other hand, $v$ cannot be $a_3$ or $a_4$ else it would have five neighbours in $X$. Hence $v$ together with $H_2$ gives a copy of $H_2^+$ in $G$.

Second suppose that $v$ is adjacent to $a_0$, to one of $a_2$ and $a_5$, and to two of $a_1$, $a_3$, $a_4$, $a_6$. By symmetry we may assume that $v$ is adjacent to $a_2$ and not to $a_5$. Then $v$ is not adjacent to $a_1$ else $va_0a_1a_2$ is a $K_4$ and $v$ is not adjacent to $a_4$ else $va_0a_1a_3a_4$ is an odd circuit in $G_{a_2}$. Thus $v$ is adjacent to $a_0$, $a_2$, $a_3$ and $a_6$. Note $v$ is not $a_4$ nor $a_5$ as $v$ does not have five neighbours in $X$. Thus $G[X \setminus \{a_1\} \cup \{v\}]$ contains a copy of $\overline{C}_7$. □

**Claim 3.18.** *Let $G$ be a locally bipartite graph containing $H_2$. If $\delta(G) > 5/9 \cdot |G|$, then $G$ contains $\overline{C}_7$.*

*Proof.* Consider a copy of $H_2$ in $G$ with vertices $X = \{a_0, a_1, \ldots, a_6\}$.




Let $U_4$ be the set of vertices with exactly four neighbours in $X$. All other vertices have at most three neighbours in $X$ so $|U_4| \geqslant 7\delta(G) - 3|G|$. Thus

$$|U_4| + |\Gamma(a_0, a_2)| \geqslant 7\delta(G) - 3|G| + 2\delta(G) - |G| = 9\delta(G) - 4|G| > |G|,$$

so $U_4$ and $\Gamma(a_0, a_2)$ are not disjoint: there is a vertex $v$ which is adjacent to both $a_0$ and $a_2$ and has four neighbours in $X$. Vertex $v$ is not adjacent to $a_1$ otherwise $va_0a_1a_2$ is a $K_4$. Also $v$ is not adjacent to $a_4$ otherwise $G_{a_2}$ contains the odd circuit $va_0a_1a_3a_4$. Note $v$ is not adjacent to both $a_5$ and $a_6$ otherwise $va_0a_5a_6$ is a $K_4$. Hence $v$ is adjacent to $a_3$. But then $v$ is not adjacent to $a_5$ else $G_{a_5}$ contains the odd circuit $va_0a_6a_4a_3$.

Hence, $v$ is adjacent to exactly $a_0$, $a_2$, $a_3$, $a_6$. In particular, $v$ is neither $a_4$ nor $a_5$. Thus $G[X \setminus \{a_1\} \cup \{v\}]$ contains $\overline{C}_7$. □

**Claim 3.19.** *Let $G$ be a locally bipartite graph containing $H_0$. Then $\delta(G) \leqslant 4/7 \cdot |G|$.*

*Proof.* Let $X$ be a set of seven vertices in $G$ with $G[X]$ containing $H_0$. Every vertex has at most four neighbours in $X$ so

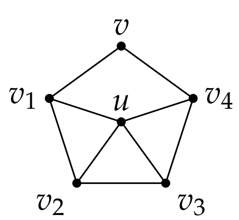
$$7\delta(G) \leqslant e(X, G) \leqslant 4|G|.$$ □

## 3.5 Ruling out Sparse Pairs being Spokes of Odd Wheels

In this section we make a start on the proof of Theorem 3.9 by ruling out the possibility that $G$ contains a sparse pair of vertices which is the spoke of an odd wheel (Corollary 3.27).

**Lemma 3.20.** *Let $G$ be an $H_0$-free, locally bipartite graph with $\delta(G) > 1/2 \cdot |G|$. Then $G$ does not contain a sparse pair $u, v$ with $uv$ being the missing spoke of a 5-wheel.*

*Proof.* By Lemma 3.8, $D_x = \{y\colon \text{the pair } x, y \text{ is dense}\}$ is independent for each vertex $x$. Suppose the conclusion does not hold: label the configuration as follows where $u, v$ is a sparse pair.

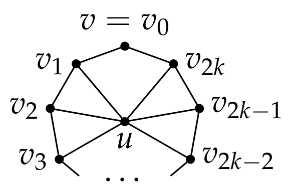



As $u, v$ is sparse, $v_1$ is not adjacent to $v_4$ and so $uv_1vv_4$ is an induced 4-cycle in $G$. By Lemma 3.7, the pair $v_1, v_4$ must be dense and so $v_4 \in D_{v_1}$. But $v_3 \in D_{v_1}$ also and so $D_{v_1}$ is not an independent set. □

We will use the following technical lemma on various occasions.

**Lemma 3.21.** *Let $G$ be a locally bipartite graph and let $u$ and $v$ be a sparse pair of vertices in $G$. Suppose that $C$ is the shortest odd cycle which both passes through $v$ and satisfies $C \setminus \{v\} \subset \Gamma(u)$ (i.e. $G[C \cup \{u\}]$ contains an odd wheel missing the spoke $uv$). Then every neighbour of $u$ has at most two neighbours in $C$ and if two, then they are two apart on $C$. In particular, $C$ is an induced cycle.*

*Proof.* Label the configuration as follows and write $v_0$ for $v$ (we consider indices modulo $2k+1$). Note $C = \{v, v_1, \ldots, v_{2k}\}$.




Consider a vertex $x$ which is adjacent to $u$. Suppose that $x$ is adjacent to two vertices in $C$ which are not two apart: $x$ is adjacent to $v_i$ and $v_{i+r}$ where $r \in \{1, 3, 4, \ldots, k\}$. Firstly, if $r = 1$, then either $G$ contains the $K_4$ $uxv_iv_{i+1}$ or $G_{u,v}$ contains an edge (if one of $v_i$ or $v_{i+1}$ is $v$) which contradicts the sparsity of $u, v$. Secondly, if $r > 1$ is odd, then $C' = xv_iv_{i+1} \cdots v_{i+r}$ is an odd cycle which is shorter than $C$. Either $C'$ is in $G_u$ (if $v \notin C'$) contradicting the local bipartness of $G$ or we have found a shorter odd cycle than $C$ which satisfies the properties of $C$ (if $v \in C'$). Finally, if $r > 2$ is even, then $C' = xv_{i+r}v_{i+r+1} \cdots v_{i-1}v_i$ is an odd cycle which is shorter than $C$. Again we either obtain an odd cycle in $G_u$ or contradict the minimality of $C$. Hence every neighbour of $u$ has at most two neighbours in $C$ and if two, then they are $v_i$, $v_{i+2}$ for some $i$.

All of $v_1, \ldots, v_{2k}$ are neighbours of $u$ so have two neighbours in $C$. This implies that $C$ is induced. □

**Lemma 3.22.** *Let $G$ be a locally bipartite graph with $\delta(G) > 8/15 \cdot |G|$ which does not contain $H_0$. Any sparse pair in $G$ that is the missing spoke of an odd wheel is the missing spoke of a 7-wheel.*

*Proof.* Consider a sparse pair $u, v$ that is the missing spoke of a $(2k+1)$-wheel. Choose the odd wheel so that $k$ is minimal. Without loss of generality, $u$ is the central vertex and $v$ is in the outer $(2k+1)$-cycle, which we call $C$. Lemma 3.20 shows that $k > 2$. We are done if we can show that $k = 3$.

By Lemma 3.21, every neighbour of $u$ has at most two neighbours in $C$. All vertices have at most $2k$ neighbours in $C$ as otherwise $G$ contains a $(2k+1)$-wheel. Hence,

$$\begin{aligned}(2k+1)\delta(G) \leqslant e(G,C) &\leqslant 2d(u) + 2k(|G| - d(u)) \\ &= 2k|G| - (2k-2)d(u) \leqslant 2k|G| - (2k-2)\delta(G),\end{aligned}$$

so

$$\frac{8}{15} < \frac{\delta(G)}{|G|} \leqslant \frac{2k}{4k-1},$$

which implies that $k < 4$. □

Thus, to rule out sparse pairs being a missing spoke of an odd wheel we only need to rule out there being a sparse pair that is the missing spoke of a 7-wheel. This is where the graph $T_0$ becomes relevant. We will prove the following.

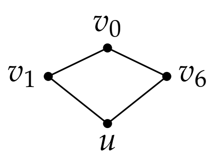
**Proposition 3.23.** *Let $G$ be a locally bipartite graph that does not contain $H_0$. If either $\delta(G) > 7/13 \cdot |G|$, or $\delta(G) > 8/15 \cdot |G|$ and $G$ does not contain $T_0$, then $G$ does not contain a sparse pair that is the missing spoke of a 7-wheel.*

*Proof.* Let $G$ be a locally bipartite graph that does not contain $H_0$ and either satisfies $\delta(G) > 7/13 \cdot |G|$, or $\delta(G) > 8/15 \cdot |G|$ and $G$ does not contain $T_0$. In a slight abuse of notation we will say that a vertex $u$ is *sparse to a cycle* $C$ if $u$ is adjacent to $|C| - 1$ vertices of $C$ and is in a sparse pair with the final vertex. We are required to show that there is no vertex $u$ and no 7-cycle $C$ with $u$ sparse to $C$.

By Lemma 3.20, there is no 5-cycle $C$ and vertex $u$ with $u$ sparse to $C$. Hence, if a vertex $u$ is sparse to a 7-cycle $C$, then, by Lemma 3.21, $C$ is an induced 7-cycle and any neighbour of $u$ has at most two neighbours in $C$.

**Claim 3.24.** *If a vertex $u$ is sparse to a 7-cycle $C = v_0v_1 \dots v_6$ with the pair $u, v_0$ sparse, then there is some vertex which has six neighbours in $C$ and is adjacent to all of $v_6$, $v_0$, $v_1$.*

*Proof of Claim.* First consider the induced 4-cycle $uv_6v_0v_1$. The pair $u, v_0$ is sparse, so $\Gamma(u, v_6, v_0) = \Gamma(v_0, v_1, u) = \varnothing$. Also $\Gamma(v_1, u, v_6) = \varnothing$ as otherwise $G_u$ contains an odd circuit. Hence all $z \notin \Gamma(v_6, v_0, v_1)$ have at most two neighbours in $\{u, v_6, v_0, v_1\}$.



Thus,

$$4\delta(G) \leqslant e(\{v_0, v_1, u, v_6\}, G) \leqslant 3|\Gamma(v_6, v_0, v_1)| + 2(|G| - |\Gamma(v_6, v_0, v_1)|),$$

so $|\Gamma(v_6, v_0, v_1)| \geqslant 4\delta(G) - 2|G|$.

If the claim is false, then all vertices in $\Gamma(v_6, v_0, v_1)$ have at most five neighbours in $C$. Also note that $\Gamma(u)$ and $\Gamma(v_6, v_0, v_1)$ are disjoint and that all vertices have at most six neighbours in $C$ (otherwise $G$ contains a 7-wheel) so,

$$\begin{aligned} 7\delta(G) \leqslant e(C, G) &\leqslant 2d(u) + 5|\Gamma(v_6, v_0, v_1)| + 6(|G| - d(u) - |\Gamma(v_6, v_0, v_1)|) \\ &= 6|G| - 4d(u) - |\Gamma(v_6, v_0, v_1)| \leqslant 6|G| - 4\delta(G) - 4\delta(G) + 2|G|, \end{aligned}$$

which contradicts $\delta(G) > 8/15 \cdot |G|$. □

**Claim 3.25.** *Let $C$ be a 7-cycle to which some vertex is sparse. Then every vertex with at least six neighbours in $C$ is sparse to $C$.*

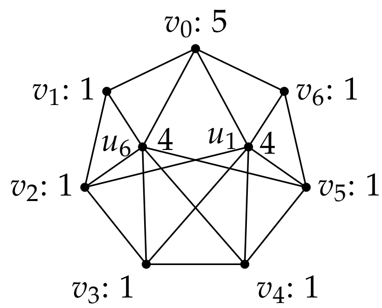
*Proof of Claim.* Let vertex $u$ be sparse to the 7-cycle $C$ and let $x$ be a vertex with at least six neighbours in $C$. Since $G$ is locally bipartite, $x$ has exactly six neighbours in $C$. Let $y$ be the vertex of $C$ to which $x$ is not adjacent. There is a vertex $u'$ which has six neighbours in $C$ and is adjacent to $y$. Indeed, if $u$ is adjacent to $y$ then take $u' = u$ and if $u$ is not adjacent to $y$, then Claim 3.24 gives the desired $u'$.

Now $\Gamma(u', x)$ contains two consecutive vertices of $C$, so the pair $u', x$ is dense. But $u'$ is adjacent to $y$ so, by Lemma 3.8, the pair $y, x$ cannot be dense. In particular, $x, y$ is sparse and so $x$ is sparse to $C$. □

Now fix a 7-cycle $C = v_0 v_1 \dots v_6$ such that there is some vertex which is sparse to $C$. Say vertex $v_i$ is *lonely* if there is some vertex $u$ which is adjacent to all of $C \setminus \{v_i\}$ – by the previous claim $u$ is sparse to $C$ and the pair $u, v_i$ is sparse. The next claim is crucial.

**Claim 3.26.** *For all $i$, $v_i$ and $v_{i+2}$ are not both lonely.*

*Proof of Claim.* Suppose for contradiction that $v_1$ and $v_6$ are both lonely: let $u_1$ and $u_6$ be sparse to $C$ with both the pairs $u_1, v_1$ and $u_6, v_6$ sparse. Lemma 3.21 shows that any neighbour of $u_1$ (or $u_6$) has at most two neighbours in $C$. Let $X = \{u_1, u_6, v_0, \dots, v_6\}$ and give weights $\omega$ to the vertices in $X$ as shown below.




For each vertex $v \in G$, let $f(v)$ be the total weight of the neighbours of $v$ in $X$. We shall show that if $v \notin \Gamma(u_6, u_1, v_0)$, then $f(v) \leqslant 10$ and if $v \in \Gamma(u_6, u_1, v_0)$, then $f(v) = 13$. Let $v$ be a vertex with $f(v) \geqslant 11$. It suffices to show that $v$ is adjacent to $u_6$, $u_1$, $v_0$ and to none of $v_1, \ldots, v_6$.

If $v$ is adjacent to neither $u_1$ nor $u_6$, then $f(v) \leqslant 11$ with equality only if $v$ is adjacent to all of $C$ which would give a 7-wheel so, in fact, $f(v) \leqslant 10$. Thus, we may assume $v$ is adjacent to $u_1$. But then, by Lemma 3.21, $v$ must have at most two neighbours in $C$. If neither of these is $v_0$, then $f(v) \leqslant 4 + 4 + 1 + 1 = 10$. Hence we may assume $v$ is adjacent to both $v_0$ and $u_1$.

Since $f(v) \geqslant 11$ and $v$ has at most two neighbours in $C$, $v$ must be adjacent to $u_6$ as well. Now, $v$ is adjacent to both $u_1$ and $v_0$ so, by Lemma 3.21, the only other possible neighbour of $v$ in $C$ is one of $v_2$ and $v_5$. However if $v$ is adjacent to $v_2$, then $G_{u_1}$ contains the odd circuit $v_0 v v_2 v_3 \ldots v_6$ while if $v$ is adjacent to $v_5$, then $G_{u_6}$ contains the odd circuit $v_0 v_1 \ldots v_5 v$. In conclusion, $v$ is adjacent to $u_1$, $u_6$, $v_0$, and no other vertices of $C$.

Thus

$$19\delta(G) \leqslant \omega(X, G) = \sum_{v \in G} f(v) \leqslant 13|\Gamma(v_0, u_1, u_6)| + 10(|G| - |\Gamma(v_0, u_1, u_6)|),$$

so

$$3|\Gamma(v_0, u_1, u_6)| \geqslant 19\delta(G) - 10|G|. \tag{3.1}$$

Any $v \in \Gamma(v_0, u_1, u_6)$ satisfies $f(v) = 13$ so has only one neighbour in $C$. Also any neighbour of $u_1$ has at most two neighbours in $C$. Hence

$$\begin{aligned} 7\delta(G) \leqslant e(C, G) &\leqslant |\Gamma(v_0, u_1, u_6)| + 2(d(u_1) - |\Gamma(v_0, u_1, u_6)|) + 6(|G| - d(u_1)) \\ &= 6|G| - 4d(u_1) - |\Gamma(v_0, u_1, u_6)|, \end{aligned}$$

so

$$|\Gamma(v_0, u_1, u_6)| \leqslant 6|G| - 11\delta(G). \tag{3.2}$$

Combining inequalities (3.1) and (3.2) gives

$$19\delta(G) - 10|G| \leqslant 3|\Gamma(v_0, u_1, u_6)| \leqslant 18|G| - 33\delta(G),$$

so $\delta(G) \leqslant 7/13 \cdot |G|$ and so $G$ is $T_0$-free.

Inequality (3.1) and $\delta(G) > 8/15 \cdot |G|$ show that $\Gamma(v_0, u_1, u_6)$ is non-empty. Let $v$ be a common neighbour of $v_0$, $u_1$, $u_6$. As $G$ is $T_0$-free, $v$ must be one of the $v_i$. But $C = \{v_0, v_1, \ldots, v_6\}$ is induced, so $v$ must be one of $v_1$, $v_6$. This means one of the edges $u_1v_1$, $u_6v_6$ is present. This gives a 7-wheel. □

We now finish the proof of Proposition 3.23. By the choice of $C$, some $v_i$ is lonely. Without loss of generality, $v_0$ is lonely. By Claim 3.26, neither $v_2$ nor $v_5$ is lonely. By Claims 3.24 and 3.25, at least one of $v_3$ and $v_4$ is lonely. By symmetry, we may assume $v_3$ is lonely. By Claim 3.26, $v_1$ is not lonely and at most one of $v_4$, $v_6$ is. Again, by symmetry, we may assume $v_6$ is not lonely. In conclusion, $v_1$, $v_2$, $v_5$, $v_6$ are all not lonely, $v_0$ and $v_3$ are lonely and $v_4$ may or may not be.

Let $U_6$ be the set of vertices with six neighbours in $C$. By Claim 3.25, any vertex in $U_6$ is sparse to $C$ so cannot be adjacent to all of $v_0$, $v_3$, $v_4$ (else some other $v_i$ is lonely). In particular,

$$U_6 \subset \overline{\Gamma(v_0)} \cup \overline{\Gamma(v_3)} \cup \overline{\Gamma(v_4)}.$$

As $v_0$ is lonely, there is a vertex $u$ that is sparse to $C$ with $u, v_0$ sparse. No two of $v_0$, $v_3$, $v_4$ are two apart on $C$ and so, by Lemma 3.21, any neighbour of $u$ is in at least two of $\overline{\Gamma(v_0)}$, $\overline{\Gamma(v_3)}$, $\overline{\Gamma(v_4)}$ and is not in $U_6$. Hence

$$|U_6| + 2d(u) \leqslant |\overline{\Gamma(v_0)}| + |\overline{\Gamma(v_3)}| + |\overline{\Gamma(v_4)}| \leqslant 3|G| - 3\delta(G),$$

and so $|U_6| \leqslant 3|G| - 5\delta(G)$. Since every neighbour of $u$ has at most two neighbours in $C$,

$$\begin{aligned} 7\delta(G) \leqslant e(C, G) &\leqslant 6|U_6| + 2d(u) + 5(|G| - |U_6| - d(u)) = 5|G| + |U_6| - 3d(u) \\ &\leqslant 5|G| + 3|G| - 5\delta(G) - 3\delta(G) = 8|G| - 8\delta(G), \end{aligned}$$

which contradicts $\delta(G) > 8/15 \cdot |G|$. □

Lemmas 3.20 and 3.22 and Proposition 3.23 together give the result we want.

**Corollary 3.27.** *Let $G$ be a locally bipartite graph that does not contain $H_0$. If either $\delta(G) > 7/13 \cdot |G|$, or $\delta(G) > 8/15 \cdot |G|$ and $G$ does not contain $T_0$, then $G$ does not contain a sparse pair which is the missing spoke of an odd wheel.*

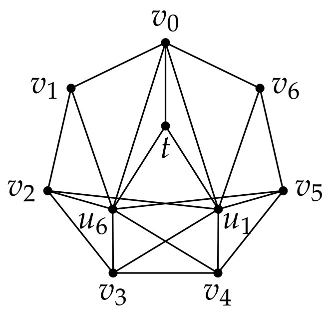
## 3.6 Finishing the Proof of Theorem 3.9

Here we will prove Theorem 3.9 which we restate for convenience.

**Theorem 3.9.** *Let $G$ be a locally bipartite graph. If $\delta(G) > 8/15 \cdot |G|$, then $G$ is either 3-colourable or contains $H_0$ or $T_0$. If $\delta(G) > 7/13 \cdot |G|$, then $G$ is either 3-colourable or contains $H_0$.*

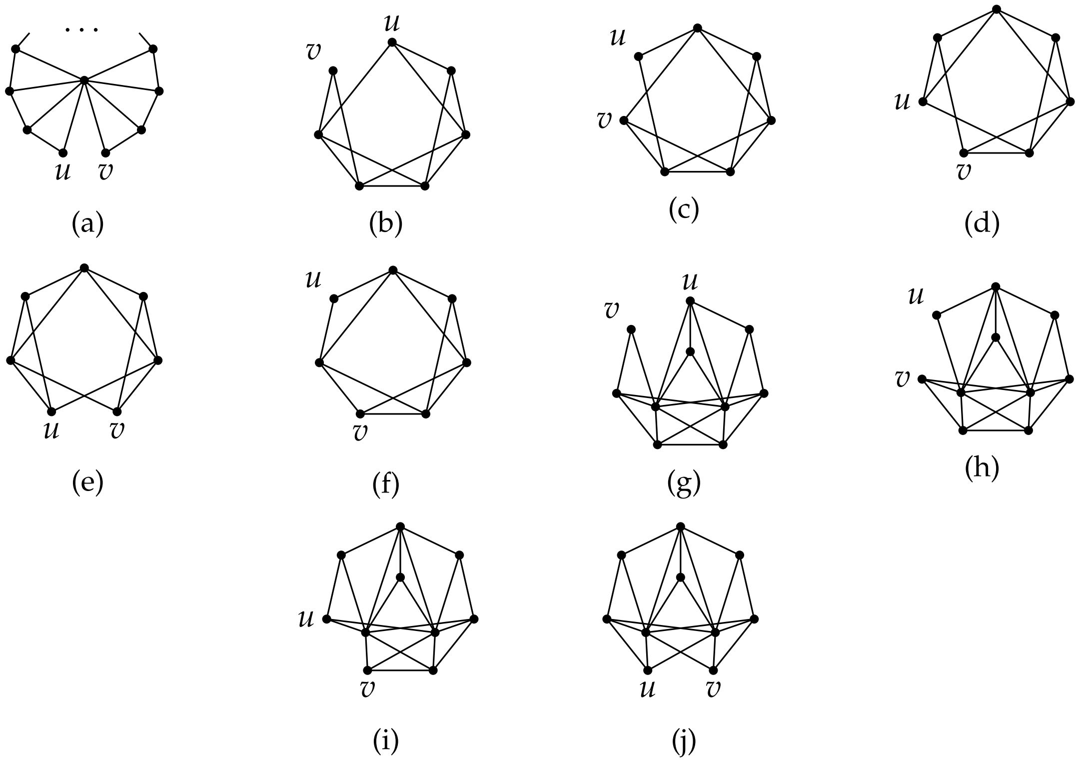
We start with a locally bipartite graph $G$ which does not contain $H_0$ and satisfies either $\delta(G) > 7/13 \cdot |G|$, or $\delta(G) > 8/15 \cdot |G|$ and $G$ does not contain $T_0$. We are required to show that $G$ is 3-colourable. We may assume that $G$ is edge-maximal: for any sparse pair $u, v$ of $G$, the addition of $uv$ to $G$ introduces an odd wheel, a copy of $H_0$ or a copy of $T_0$. By Theorem 3.10, the addition of $uv$ to $G$ introduces an odd wheel, a copy of $H_2$ (since $G$ itself does not contain $H_2$) or a copy of $T_0$.

Firstly, if the addition of $uv$ introduces an odd wheel, then, by Corollary 3.27, $uv$ must be a rim of that wheel – this case is depicted in Figure 3.5a below. Secondly, if the addition of $uv$ introduces a copy of $H_2$, then that copy of $H_2$ less the edge $uv$ must not contain $H_0$ – this case is depicted in Figures 3.5b to 3.5f below. Finally, suppose the addition of $uv$ introduces a copy of $T_0$ (but not an odd wheel nor a copy of $H_0$). Label this copy of $T_0$ in $G + uv$ as follows.




Note that $G + uv$ is locally bipartite and does not contain $H_0$ so, by Lemma 3.8, for any vertex $x$, $D_x = \{y\colon \text{the pair } x, y \text{ is dense}\}$ is an independent set. In $G + uv$, $t \in D_{v_1}$ and $t$ is adjacent to $u_1$ so the pair $u_1, v_1$ is not dense. Also $u_1v_1$ is not an edge (else $G + uv$ contains a 7-wheel centred at $u_1$), so the pair $u_1, v_1$ is sparse in $G + uv$. Therefore, $u_1, v_1$ is a sparse pair in $G$. Now, by Corollary 3.27, $G$ does not contain an odd wheel with a sparse spoke so $uv$ must either be one of the edges $v_iv_{i+1}$ or one of the edges $u_1v_i$. Similarly $uv$ must either be one of the edges $v_iv_{i+1}$ or one of the edges $u_6v_i$. Thus, in fact, $uv$ must be one of the edges $v_iv_{i+1}$ and by symmetry we may take $i = 0, 1, 2, 3$ – this case is depicted in Figures 3.5g to 3.5j below.

Thus, in $G$, any sparse pair $u, v$ must appear in one of the following configurations (with the labels of $u$ and $v$ possibly swapped).



(a) (b) (c) (d) (e) (f) (g) (h) (i) (j)

Figure 3.5

We will now consider a largest independent set in $G$: an independent set $I$ of size $\alpha(G)$. We will shortly show that all vertices are either in $I$ or adjacent to all of $I$. Recall for a vertex $u$ that

$$D_u = \{v\colon \text{the pair } u, v \text{ is dense}\}.$$

By Lemma 3.6, for every $u \in I$, $I \subset D_u \cup \{u\}$. We now show there is set equality.

**Proposition 3.28.** *For every $u \in I$, $I = D_u \cup \{u\}$*

*Proof.* By Lemma 3.8 and the definition of dense, $D_u \cup \{u\}$ is an independent set. However, it contains the maximal independent set $I$, so must equal it. □

The following definition will be helpful.

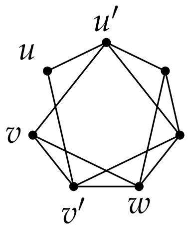
**Definition 3.29** (quasidense)**.** *A pair of vertices $u, v$ is* ***quasidense*** *if there is a sequence of vertices $u = d_1, d_2, \dots, d_k, d_{k+1} = v$ such that all pairs $d_i, d_{i+1}$ are dense ($i = 1, 2, \dots, k$).*

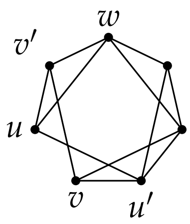
Proposition 3.28 immediately implies that if the pair $u, v$ is quasidense and $u \in I$, then $v \in I$ as well.

**Proposition 3.30.** *Every vertex of $G$ is either in $I$ or adjacent to all of $I$.*

*Proof.* Fix a vertex $u \in I$ and let $v$ be any other vertex which is not adjacent to $u$. It suffices to show that $v \in I$. If the pair $u, v$ is (quasi)dense, then $v \in I$, so we may assume that $u, v$ is sparse (and not quasidense). Thus $u, v$ appears in one of the configurations given in Figure 3.5 (with labels $u$ and $v$ possibly swapped). However in each of Figures 3.5a, 3.5b, 3.5e and 3.5g to 3.5j the pair $u, v$ is quasidense. Hence we may assume that $u, v$ appear in one of Figures 3.5c, 3.5d and 3.5f.

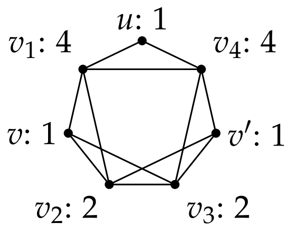
We consider Figures 3.5c and 3.5d together. For ease we label some more of the vertices as follows.




In both cases, the pair $u', w$ is dense and so, by Lemma 3.8, the pair $u', v'$ is not dense. However, $u'v'$ is not an edge, as the pair $u, v$ is sparse, and so $u', v'$ is a sparse pair. But then $uu'vv'$ is an induced 4-cycle in which both non-edges are sparse which contradicts Lemma 3.7.

Finally we consider Figure 3.5f which we label as follows. Let $X = \{u, v, v', v_1, v_2, v_3, v_4\}$ and give weights $\omega$ to the vertices of $X$ as shown.




The pair $v, v'$ is dense, so, if $u, v'$ is quasidense, then $u, v$ is quasidense, a contradiction. Also, as $G$ is $H_0$-free, $uv'$ is not an edge. Hence the pair $u, v'$ is sparse and not quasidense. Now $v, v'$ is dense and so, by Lemma 3.8, the pair $vv_4$ is not. But $vv_4$ is not an edge (else $u, v$ is dense), so $v, v_4$ is a sparse pair. Similarly, $v', v_1$ is a sparse pair. To summarise, the pairs $u, v$ and $u, v'$ are sparse and not quasidense and the pairs $v, v_4$ and $v', v_1$ are sparse. It follows that $G[X]$ contains no more edges than shown.

If a vertex $x$ has five neighbours in $X$, then it is adjacent to three consecutive vertices round the 7-cycle, so $x$ is either adjacent to a triangle or to all of $u$, $v_1$, $v$ or to all of $v'$,

$v_4, u$. The first gives a $K_4$ while the latter two contradict the pairs $u, v$ and $u, v'$ being sparse. Hence all vertices have at most four neighbours in $X$.

For each vertex $x$ in $G$, let $f(x)$ be the total weight of the neighbours of $x$ in $X$. Now

$$\sum_{x \in G} f(x) = \omega(X, G) \geqslant 15\delta(G) > 8|G|,$$

so some vertex $x$ has $f(x) \geqslant 9$. All vertices of $X$ have $f$ value at most eight, so $x \notin X$. As $x$ has at most four neighbours in $X$, either $x$ is adjacent to both $v_1$ and $v_4$ or $x$ is adjacent to exactly one of $v_1, v_4$ and both of $v_2, v_3$.

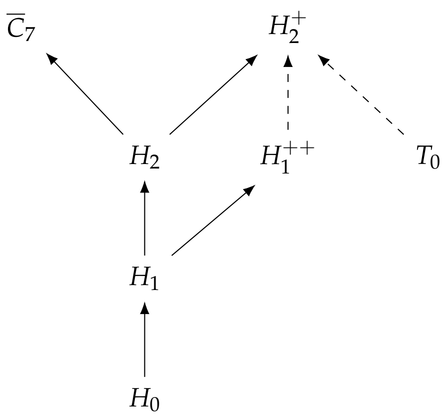
First suppose $x$ is adjacent to both $v_1$ and $v_4$. As $v, v_4$ is sparse, $x$ is not adjacent $v$. Similarly $x$ is not adjacent to $v'$. If $x$ is adjacent to $v_2$, then $x, v$ is dense. But also $u, x$ is dense (edge $v_1v_4$), so $u, v$ is quasidense, a contradiction. Hence $x$ is not adjacent to $v_2$. Similarly $x$ is not adjacent to $v_3$, and so $f(x) = 8$, a contradiction.

In the second case, we may assume, by symmetry, that $x$ is adjacent to $v_1, v_2, v_3$ but not to $v_4$. Then $x, v$ and $x, v'$ are both dense pairs (edge $v_2v_3$) so $x$ is adjacent to neither $v$ or $v'$. Finally, if $x$ is adjacent to $u$, then $x, v_4$ is dense (edge $uv_1$). But then the edge $v_4v'$ is in $D_x$, contradicting Lemma 3.8. Hence $f(x) = 8$, a contradiction. □

*Proof of Theorem 3.9.* Let $u \in I$. Proposition 3.30 gives $G[V(G) \setminus I] = G_u$, so $G[V(G) \setminus I]$ is 2-colourable. Using a third colour for the independent set $I$ gives a 3-colouring of $G$. □

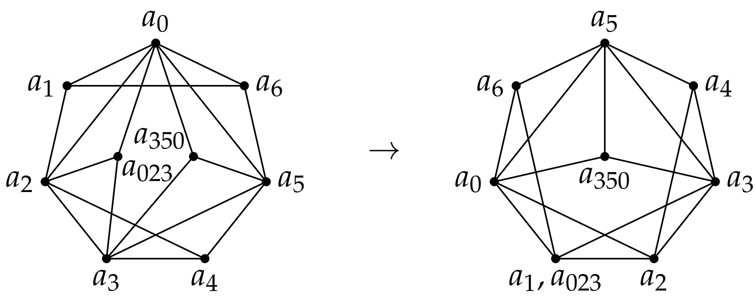
## 3.7 Verifying Figure 3.2

This section is a technical one verifying Figure 3.2, which, for convenience, we display again here.

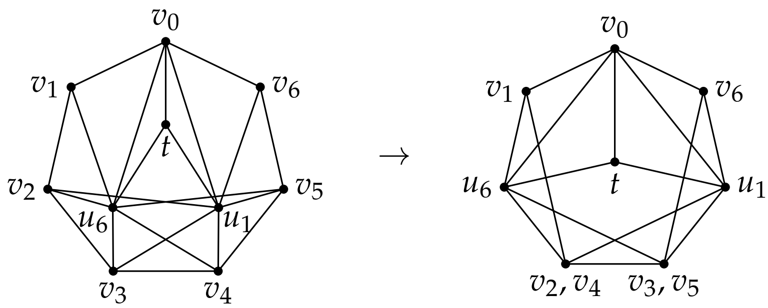



The reader will recall that full arrows represent containment and dashed arrows represent homomorphisms. Furthermore, $H$ is homomorphic to $G$ in the diagram if there is a sequence of arrows starting at $H$ and ending at $G$.

All the containments are clear. The following figure gives a homomorphism from $H_1^{++}$ to $H_2^{+}$: the left diagram is a labelling of the vertices of $H_1^{++}$ and the right diagram shows the images of those vertices under the map.




The following figures gives a homomorphism from $T_0$ to $H_2^{+}$: the left diagram is a labelling of the vertices of $T_0$ and the right diagram shows the images of those vertices under the map.




In particular, all arrows in Figure 3.2 are correct. We need to show that further arrows cannot be added. There are will be subtleties in our notation that we now elucidate. Given a homomorphism $\varphi\colon H \to G$, we say $\varphi$ is surjective or injective if the map $\varphi\colon V(H) \to V(G)$ is surjective or injective, respectively. Note that $\varphi$ being injective implies that $H$ is actually a subgraph of $G$. By $\varphi(H)$ we mean the graph on vertex set $\varphi(V(H))$ and edge set $\varphi(E(H))$. In particular, this is a spanning subgraph of $G[\varphi(V(H))]$ but it may not have all the edges of $G[\varphi(V(H))]$. We make frequent use of the fact that $\chi(\varphi(H)) \geqslant \chi(H)$.

We first deal with left-hand side ($H_0 \to H_1 \to H_2 \to \overline{C}_7$) of the figure: we need to show that $\overline{C}_7 \nrightarrow H_2$, $H_2 \nrightarrow H_1$, and $H_1 \nrightarrow H_0$. The arguments are very similar, making use of the fact that $H_0$, $H_1$, $H_2$, and $\overline{C}_7$ are all vertex-critical 4-chromatic graphs on seven vertices, so we only give the explicit proof for $H_2 \nrightarrow H_1$.

**Proposition 3.31.** *The graph $H_2$ is not homomorphic to $H_1$.*

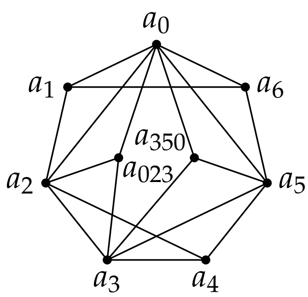
*Proof.* Suppose there is a homomorphism $\varphi\colon H_2 \to H_1$. Then $\chi(\varphi(H_2)) \geqslant \chi(H_2) = 4$. Now $H_1$ is a vertex-critical 4-chromatic graph, so $\varphi$ is surjective. But $H_1$ and $H_2$ both have seven vertices so $\varphi$ is injective. That is, $H_1$ must contain a copy of $H_2$, which is absurd as $e(H_1) < e(H_2)$. $\square$

We now know that the left-hand side of Figure 3.2 is correct and consider how $H_2^+$ relates to it. It suffices to show that $H_2^+ \nrightarrow \overline{C}_7$ and $\overline{C}_7 \nrightarrow H_2^+$ (note that $H_2^+ \nrightarrow \overline{C}_7$ implies $H_2^+ \nrightarrow H_2, H_1, H_0$). That $H_2^+$ is not homomorphic to $\overline{C}_7$ and vice versa follows from Lemma 3.4 – both $\overline{C}_7$ and $H_2^+$ are edge-maximal locally bipartite graphs and neither is a subgraph of the other ($\overline{C}_7$ has fewer vertices than $H_2^+$ and $H_2^+$ does not have seven vertices all of degree at least four).

Next we relate $H_1^{++}$ to the diagram. It suffices to show that $H_1^{++} \nrightarrow \overline{C}_7$ and $H_2 \nrightarrow H_1^{++}$ (note that $H_1^{++} \nrightarrow \overline{C}_7$ implies $H_1^{++} \nrightarrow H_2, H_1, H_0$ while $H_2 \nrightarrow H_1^{++}$ implies $\overline{C}_7 \nrightarrow H_1^{++}$ and $H_2^+ \nrightarrow H_1^{++}$).

**Proposition 3.32.** *The graph $H_1^{++}$ is not homomorphic to $\overline{C}_7$.*

*Proof.* Suppose there is a homomorphism $\varphi\colon H_1^{++} \to \overline{C}_7$. Label the copy of $H_1^{++}$ as shown below and let $A = \{a_0, a_1, \ldots, a_6\}$ so $H_1^{++}[A]$ is a copy of $H_1$. Note that $\chi(\varphi(H_1^{++}[A])) \geqslant \chi(H_1^{++}[A]) = 4$ and $\overline{C}_7$ is a vertex-critical 4-chromatic graph, so the restriction of $\varphi$ to $A$ is a surjection onto $\overline{C}_7$ and so $\varphi(a_0), \varphi(a_1), \ldots, \varphi(a_6)$ are all distinct.




Now $a_0$ has degree 6 while $\varphi(a_0) \in \overline{C}_7$ only has degree 4. The four neighbours of $\varphi(a_0)$ are $\varphi(a_1)$, $\varphi(a_2)$, $\varphi(a_5)$, $\varphi(a_6)$ and so $\varphi(a_{023})$ is one of these. Also $a_2$ has degree 5 while $\varphi(a_2)$ only has degree 4. The four neighbours of $\varphi(a_2)$ are $\varphi(a_0)$, $\varphi(a_1)$, $\varphi(a_3)$, $\varphi(a_4)$ and so $\varphi(a_{023})$ is one of these. Hence $\varphi(a_{023}) = \varphi(a_1)$. Similarly, considering the neighbourhoods of $a_0$ and $a_5$ shows that $\varphi(a_{350}) = \varphi(a_6)$. Then $\Gamma(\varphi(a_3))$ contains

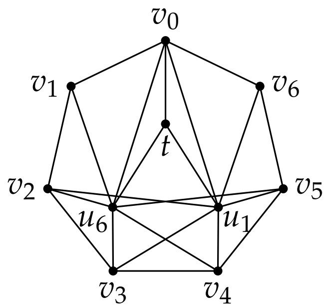
$$\{\varphi(a_2), \varphi(a_{023}), \varphi(a_{350}), \varphi(a_4), \varphi(a_5)\} = \{\varphi(a_2), \varphi(a_1), \varphi(a_6), \varphi(a_4), \varphi(a_5)\},$$

which has size 5. This contradicts the 4-regularity of $\overline{C}_7$. $\square$

**Proposition 3.33.** *The graph $H_2$ is not homomorphic to $H_1^{++}$.*

*Proof.* Suppose there is a homomorphism $\varphi\colon H_2 \to H_1^{++}$. Now $\chi(\varphi(H_2)) \geqslant \chi(H_2) = 4$ and any 6-vertex subgraph of $H_1^{++}$ is 3-colourable (it is homomorphic to some 6-vertex subgraph of $H_2^{+}$) so $\varphi$ must be injective. Thus $H_1^{++}$ contains $H_2$. But $H_1^{++}$ only has 4 vertices of degree at least 4 while $H_2$ has 5 vertices of degree 4. □

Finally we relate $T_0$ to the diagram. It suffices to show that $H_0 \nrightarrow T_0$, $T_0 \nrightarrow \overline{C}_7$, and $T_0 \nrightarrow H_1^{++}$ (note that $H_0 \nrightarrow T_0$ implies that no other graph in the diagram is homomorphic to $T_0$ while $T_0 \nrightarrow \overline{C}_7$ implies that $T_0 \nrightarrow H_0, H_1, H_2$). We use the following labelling of $T_0$ in all three proofs.



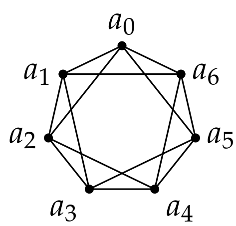

**Proposition 3.34.** *The graph $H_0$ is not homomorphic to $T_0$.*

*Proof.* We first claim that any 7-vertex subgraph of $T_0$ is 3-colourable. Let $F$ be a 7-vertex subgraph of $T_0$. If $F$ contains all the $v_i$, then $F$ is a subgraph of a 7-cycle and so is 3-colourable. Otherwise $F$ is a subgraph of $T_0 - v_i$ for some $i$. This graph is 3-colourable: 2-colour the remaining $v_j$ with colours 1 and 2, give $u_1$ and $u_6$ colour 3 and then give $t$ colour 1 or 2 (opposite to the colour of $v_0$ if it is present).

Suppose there is a homomorphism $\varphi\colon H_0 \to T_0$. Then $\varphi(H_0)$ is a subgraph of $T_0$ with at most 7 vertices, so is 3-colourable. But then $3 \geqslant \chi(\varphi(H_0)) \geqslant \chi(H_0) = 4$. □

**Proposition 3.35.** *The graph $T_0$ is not homomorphic to $\overline{C}_7$.*

*Proof.* Suppose $\varphi\colon T_0 \to \overline{C}_7$ is a homomorphism. Label the copy of $\overline{C}_7$ as follows.



Without loss of generality we may assume $\varphi(u_1) = a_0$. The common neighbourhood $\Gamma(u_1, u_6)$ contains the edge $tv_0$, so $\Gamma(\varphi(u_1), \varphi(u_6))$ contains an edge and so $\varphi(u_6) \in \{a_0, a_3, a_4\}$. By symmetry, we may assume $\varphi(u_6) \in \{a_0, a_3\}$.

First suppose that $\varphi(u_6) = a_0$. Then

$$\varphi(\{v_0, v_1, \dots, v_6\}) \subset \varphi(\Gamma(u_1) \cup \Gamma(u_6)) \subset \Gamma(\varphi(u_1)) \cup \Gamma(\varphi(u_6)) = \Gamma(a_0).$$

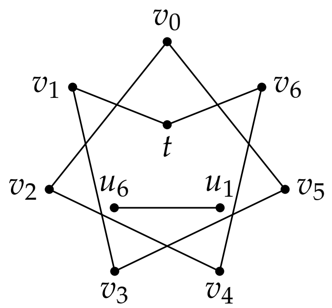
However, $v_0 v_1 \dots v_6$ form a 7-cycle which is 3-chromatic, while $\Gamma(a_0)$ is a path of length 3 (which is bipartite).

Now suppose that $\varphi(u_6) = a_3$. The edge $tv_0$ is in $\Gamma(u_1, u_6)$ so $\varphi(t)\varphi(v_0)$ must be an edge in $\Gamma(a_0, a_3)$. In particular, $\{\varphi(t), \varphi(v_0)\} = \{a_1, a_2\}$. By symmetry we may assume that $\varphi(v_0) = a_1$. Now $v_1 \in \Gamma(v_0, u_6)$, so $\varphi(v_1) \in \Gamma(a_1, a_3)$ and so $\varphi(v_1) = a_2$. Next $v_2 \in \Gamma(u_1, u_6, v_1)$, so $\varphi(v_2) \in \Gamma(a_0, a_3, a_2)$ and so $\varphi(v_2) = a_1$. Working in this way round the outer 7-cycle gives $\varphi(v_3) = a_2$, $\varphi(v_4) = a_1$ and $\varphi(v_5) = a_2$. Finally $v_6 \in \Gamma(u_1, v_0, v_5)$ and so $\varphi(v_6) \in \Gamma(a_0, a_1, a_2) = \varnothing$, which is a contradiction. □

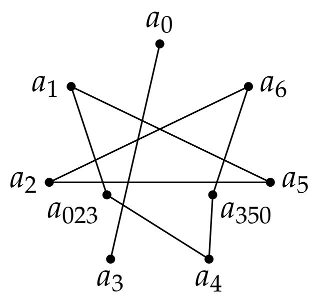
**Proposition 3.36.** *The graph $T_0$ is not homomorphic to $H_1^{++}$.*

*Proof.* Suppose $\varphi\colon T_0 \to H_1^{++}$ is a homomorphism. If $x, y$ is a dense pair (see Definition 3.5) of vertices in $T_0$, then $\Gamma(x, y)$ contains an edge, so $\Gamma(\varphi(x), \varphi(y))$ contains an edge and so either $\varphi(x) = \varphi(y)$ or $\varphi(x), \varphi(y)$ is a dense pair in $H_1^{++}$.

For a graph $G$, let $\mathcal{D}G$ be the graph with vertex set $V(G)$ and with vertices $x$ and $y$ adjacent if $x, y$ is a dense pair in $G$. The previous paragraph shows that $\varphi$ maps any connected component of $\mathcal{D}T_0$ to one in $\mathcal{D}H_1^{++}$. The graphs $\mathcal{D}T_0$ and $\mathcal{D}H_1^{++}$ are displayed below (we have used the same labelling of the vertices of $H_1^{++}$ as in Proposition 3.32).




$\mathcal{D}T_0$




$\mathcal{D}H_1^{++}$

Let $C$ be the 8-cycle $v_0 v_2 v_4 v_6 t v_1 v_3 v_5$ of $\mathcal{D}T_0$ and so $\varphi(C)$ is connected in $\mathcal{D}H_1^{++}$. If $\varphi(C)$ meets $\{a_0, a_3\}$, then $|\varphi(C)| \leqslant 2$ and so $|\varphi(T_0)| \leqslant 4$ while $\chi(\varphi(T_0)) \geqslant \chi(T_0) = 4$ so $\varphi(T_0)$ is a 4-clique. However, $H_1^{++}$ is $K_4$-free, and so $\varphi(C) \subset H_1^{++} - \{a_0, a_3\}$.

Now $H_1^{++} - a_0$ and $H_1^{++} - a_3$ are both 3-colourable (they are both 2-degenerate) and $\chi(\varphi(T_0)) \geqslant 4$, so $a_0, a_3 \in \varphi(T_0)$. In particular, $\varphi(\{u_1, u_6\}) = \{a_0, a_3\}$. By symmetry we may assume that $\varphi(u_1) = a_0$ and $\varphi(u_6) = a_3$.

In $T_0$, the path $v_2v_3v_4v_5$ lies in the common neighbourhood of $u_1$ and $u_6$. While, in $H_1^{++}$, the common neighbourhood of $a_0 = \varphi(u_1)$, $a_3 = \varphi(u_6)$ consists of two disconnected edges $a_2a_{023}$ and $a_5a_{350}$. Thus $\{\varphi(v_2), \varphi(v_5)\}$ is either $\{a_2, a_{023}\}$ or $\{a_5, a_{350}\}$.

Back in $\mathcal{D}T_0$, $v_2v_0v_5$ is a path, so $\varphi(v_2)\varphi(v_0)\varphi(v_5)$ is a single vertex, edge, or path in $\mathcal{D}H_1^{++}$. But this is inconsistent with $\{\varphi(v_2), \varphi(v_5)\}$ being either $\{a_2, a_{023}\}$ or $\{a_5, a_{350}\}$. □

## 3.8 Homomorphism results

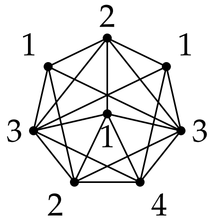
In this section, we will prove Theorems 3.11 to 3.13, which we restate here for convenience.

**Theorem 3.11.** *Let $G$ be a locally bipartite graph. If $\delta(G) > 6/11 \cdot |G|$ and $G$ contains $\overline{C}_7$, then $G$ is homomorphic to $\overline{C}_7$.*

**Theorem 3.12.** *Let $G$ be a locally bipartite graph. If $\delta(G) > 6/11 \cdot |G|$, then $G$ is 4-colourable.*

**Theorem 3.13.** *There is an $\varepsilon > 0$ such that if $G$ is a locally bipartite graph with $\delta(G) > (5/9 - \varepsilon) \cdot |G|$ and $G$ does not contain $\overline{C}_7$, then $G$ is homomorphic to $H_2^+$.*

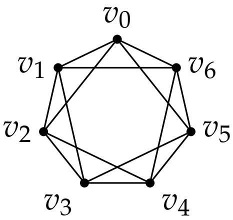
The proof of Theorem 3.11 (which appears in § 3.8.1) takes a copy of $\overline{C}_7$ in $G$ and builds structure around it, focussing initially on those vertices with four neighbours in the copy of $\overline{C}_7$ (of which there are many – more than $9/11 \cdot |G|$ in fact) and then tacking the rest onto these. The proof of Theorem 3.12 (which appears in § 3.8.2) is essentially similar but longer. In place of a copy of $\overline{C}_7$ we take a copy of $H_2^+$ (if one of these is not present, then by Theorem 3.9 and Theorem 3.10, $G$ is either 3-colourable or contains $\overline{C}_7$ and so we are done by Theorem 3.11). Around this copy of $H_2^+$ structure is built in an analogous way to the proof of Theorem 3.11 with the aim of showing that $G$ is homomorphic to $H_2^+$ (as we believe it is). We will not complete this endeavour fully but will show that $G$ is homomorphic to the following graph, which is $H_2^+$ with four extra edges.




Thankfully, this graph is 4-colourable (colouring shown in the diagram) and so we have Theorem 3.12. Finally, the proof of Theorem 3.13 (which appears in § 3.8.3) uses all the machinery developed in the proof of Theorem 3.12 and makes use of taking $\varepsilon$ sufficiently small so that the overall structure of $G$ is very similar to that of the weighted $H_2^+$ shown in Figure 3.3 on page 33 (which had minimum degree 5/9).

### 3.8.1 Proof of Theorem 3.11

In this subsection we prove Theorem 3.11. Fix a locally bipartite graph $G$ with $\delta(G) > 6/11 \cdot |G|$ that contains a copy of $\overline{C}_7$, which we label as follows. We will always consider indices modulo seven.




Let

$$\begin{aligned} D &= \{x \in V(G) \colon x \text{ is adjacent to four of } v_0, v_1, \ldots, v_6\}, \\ R &= \{x \in V(G) \colon x \text{ is adjacent to at most three of } v_0, v_1, \ldots, v_6\}. \end{aligned}$$

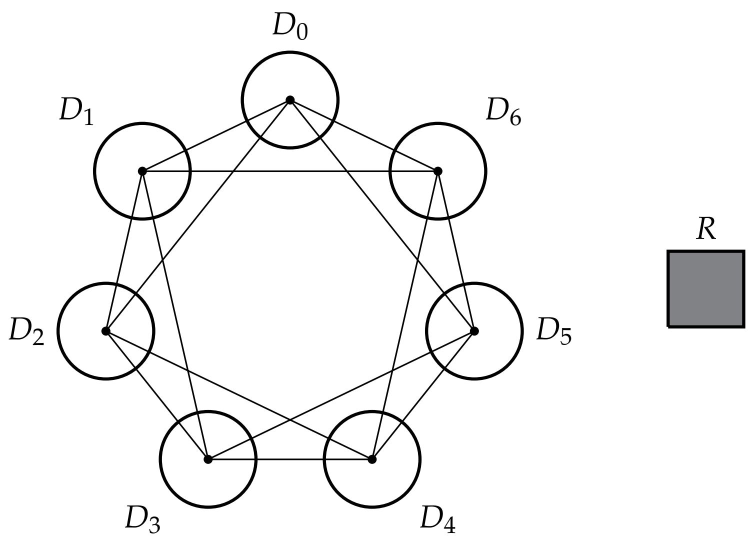
Observation 3.3 says that no vertex is adjacent to five of the $v_i$, so $D \cup R$ partitions $V(G)$. More precisely, note that no vertex is adjacent to three consecutive $v_i$ (otherwise there is a $K_4$) nor to all of $v_{i-2}, v_i, v_{i+2}$ (otherwise there is a 5-wheel centred at $v_i$). In particular, if we let

$$D_i = \Gamma(v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}),$$

then $D_0 \cup D_1 \cup \cdots \cup D_6$ partitions $D$. We have a simple upper bound for the size of $R$.

$$\begin{aligned} &7\delta(G) \leqslant e(\{v_0, v_1, \ldots, v_6\}, G) \leqslant 4|D| + 3|R| = 4|G| - |R|, \\ &\Rightarrow |R| \leqslant 4|G| - 7\delta(G). \end{aligned} \tag{3.3}$$

Also note that $D_i \cup D_{i+3}$ is independent for all $i$: if there is an edge $dd'$ inside $D_i \cup D_{i+3}$, then $dv_{i+1}v_{i+2}d'$ is a $K_4$. In particular, $G[D]$ is homomorphic to $\overline{C}_7$. Our aim is to get a handle on $R$.




We will use the following lemma frequently.

**Lemma 3.37.** *Let $X \subset V(G)$ be a set of four vertices. Either there is $x \in R$ adjacent to all of $X$ or there is $x \in D$ with at least three neighbours in $X$.*

*Proof.* Using inequality (3.3) and $\delta(G) > 6/11 \cdot |G|$, we have

$$e(X, G) \geqslant 4\delta(G) > 6|G| - 7\delta(G) \geqslant 2|G| + |R| = 2|D| + 3|R|.$$

But $D \cup R$ partitions $V(G)$, so either some vertex in $D$ has more than two neighbours in $X$ or some vertex in $R$ has more than three neighbours in $X$. □

Our first two claims show that the collections of $v_i$ to which vertices can be adjacent are similar to the collections of the $D_i$ in which vertices can have neighbours.

**Claim 3.38.** *For all $i$, no vertex has a neighbour in each of $D_{i-1}$, $D_i$, $D_{i+1}$.*

*Proof.* If not, without loss of generality, we may choose $d_6$, $d_0$, $d_1$ in $D_6$, $D_0$, $D_1$ respectively with common neighbour $u$ such that $e(\{d_6, d_0\}) + e(\{d_0, d_1\})$ is maximal. We now apply Lemma 3.37 to $\{u, d_6, d_0, d_1\}$.

Suppose some $x$ is adjacent to all of $u$, $d_6$, $d_0$, $d_1$. Now apply Lemma 3.37 to $X = \{u, x, d_6, d_1\}$: as $uxd_6$ and $uxd_1$ are triangles, no vertex is adjacent to all of $X$ and furthermore, any vertex with three neighbours in $X$ must be adjacent to both $d_6$ and $d_1$.

In particular, some $d' \in D$ is adjacent to $d_6$, $d_1$ and to one of $u$, $x$. But then $d' \in D_0$, so, in our choice of $d_6$, $d_0$, $d_1$, $u$ at the start, we could swap $d'$ for $d_0$ and $u$ for whichever of $u$ and $x$ is adjacent to $d'$. This contradicts the maximality unless $d_0$ is adjacent to both $d_6$ and $d_1$. But then $d_0 d_1 u x$ is a $K_4$.

Hence, in fact, there is some $d \in D$ adjacent to three of $u$, $d_6$, $d_0$, $d_1$. No vertex in $D$ is adjacent to all of $d_6$, $d_0$, $d_1$, so $d$ is adjacent to $u$. By symmetry we may assume $d$ is adjacent to $d_6$. If $d$ is adjacent to $d_0$ as well, then (as $d \in D$) $d$ is adjacent to both $v_6$ and $v_0$. But then $u d_6 v_0 v_6 d_0$ is an odd circuit in $G_d$.

Thus $d$ is adjacent to $u, d_6$, and $d_1$. But then $d \in D_0$ and so $u d_6 v_1 v_6 d_1$ is an odd circuit in $G_d$. □

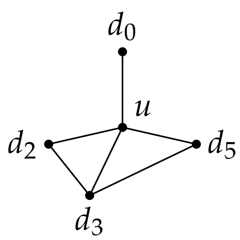
**Claim 3.39.** *For all $i$, no vertex has a neighbour in each of $D_{i-2}$, $D_i$, $D_{i+2}$.*

*Proof.* If not, without loss of generality we may choose $d_5$, $d_0$, $d_2$ in $D_5$, $D_0$, $D_2$ respectively with common neighbour $u$ such that $e(\{d_5, d_0\}) + e(\{d_0, d_2\})$ is maximal. No vertex in $D$ has a neighbour in each of $D_5$, $D_0$ and $D_2$, so $u \in R$. Apply Lemma 3.37 to $\{u, d_5, d_0, d_2\}$.

Suppose some $x$ is adjacent to all of $u$, $d_5$, $d_0$, $d_2$. Now apply Lemma 3.37 to $\{u, x, d_0, d_2\}$: as $u x d_0$ and $u x d_2$ are triangles, there must be some $d' \in D$ is adjacent to $d_0$, $d_2$ and one of $u$, $x$. But then $d' \in D_1$ and so one of $u$, $x$ has a neighbour in each of $D_0$, $D_1$, and $D_2$ which contradicts Claim 3.38.

Hence, there is some $d \in D$ adjacent to three of $u$, $d_5$, $d_0$, $d_2$. No vertex in $D$ is adjacent to all of $d_5$, $d_0$, $d_2$ so $d$ is adjacent to $u$. By symmetry, we may assume $d$ is adjacent to $d_2$. If $d$ is adjacent to $d_0$ as well, then $d \in D_1$, so $u$ has a neighbour in each of $D_0$, $D_1$, $D_2$ contradicting Claim 3.38.

Thus $d$ is adjacent to $u$, $d_5$, and $d_2$, so $d \in D_0 \cup D_3 \cup D_4$. If $d \in D_0$, then $u d_5 v_6 v_1 d_2$ is an odd circuit in $G_d$. Hence, we may assume by symmetry that $d \in D_3$. Write $d_3$ for $d$.




We now show there is some $d_0' \in D_0$ adjacent to both $u$ and $d_5$. Apply Lemma 3.37 to $\{u, d_5, v_6, d_0\}$: by Claim 3.38, no vertex is adjacent to all of $d_5$, $v_6$, $d_0$ so there is $d'' \in D$ adjacent to $u$ and to two of $d_5$, $v_6$, $d_0$.

- If $d''$ is adjacent to $d_5$ and $d_0$, then $d'' \in D_6$, so $u$ has a neighbour in each of $D_5$, $D_6$, $D_0$, contrary to Claim 3.38.
- If $d''$ is adjacent to $d_5$ and $v_6$, then $d'' \in D_4 \cup D_0$. But if $d'' \in D_4$, then $u$ has a neighbour in each of $D_2$, $D_3$, $D_4$, contrary to Claim 3.38, so $d'' \in D_0$. We may take $d_0' = d''$.
- If $d''$ is adjacent to $v_6$ and $d_0$, then $d'' \in D_5 \cup D_1$. But if $d'' \in D_1$, then $u$ has a neighbour in each of $D_0$, $D_1$, $D_2$, contrary to Claim 3.38, so $d'' \in D_5$. By the maximality at the start we must have $d_5$ adjacent to $d_0$. We may take $d_0' = d_0$.

Thus there is some $d_0' \in D_0$ adjacent to both $u$ and $d_5$. But then $d_0' u d_3 v_4 v_6$ is an odd circuit in $G_{d_5}$. □

From the previous two claims it follows that for every vertex $v$ there is an $i$ such that

$$\Gamma(v) \cap D \subset \Gamma(v_i) \cap D = D_{i-2} \cup D_{i-1} \cup D_{i+1} \cup D_{i+2}.$$

For $i = 0, 1, \ldots, 6$ choose

$$R_i \subset \{v \in R \colon \Gamma(v) \cap D \subset \Gamma(v_i) \cap D\},$$

so that $R_0 \cup R_1 \cup \cdots \cup R_6$ is a partition of $R$. There may be some flexibility in the choice of the $R_i$ (e.g. if $\Gamma(v) \cap D \subset D_0 \cup D_3 \cup D_4$, then we could take $v$ in $R_2$ or $R_5$) – we will make use of this later. For now we just take any arbitrary choice. Note, by definition, that

$$e(R_i, D_i \cup D_{i-3} \cup D_{i+3}) = 0.$$

For each $i$, let $T_i = D_i \cup R_i$ – note that these partition $V(G)$. We can give a lower bound for the size of $T_i$. For each $i$,

$$\begin{aligned} d(v_{i-1}) + d(v_{i+1}) &= |D_i| + |D| + |R \cap \Gamma(v_{i-1})| + |R \cap \Gamma(v_{i+1})| \\ &= |D_i| + |D| + |R \cap (\Gamma(v_{i-1}) \cup \Gamma(v_{i+1}))| + |R \cap \Gamma(v_{i-1}, v_{i+1})| \\ &\leqslant |D_i| + |D| + |R| + |R_i| = |G| + |T_i|, \end{aligned}$$

so

$$|T_i| \geqslant 2\delta(G) - |G|. \tag{3.4}$$

We will eventually show that $T_i \cup T_{i+3}$ is independent for all $i$ and so the map sending all vertices in $T_i$ to $v_i$ is a homomorphism from $G$ to $\overline{C}_7$.

**Claim 3.40.** *Every $d \in D_i$ and $u \in T_{i+1}$ have a common neighbour in $D$. Similarly, every $d \in D_i$ and $u \in T_{i-1}$ have a common neighbour in $D$.*

*Proof.* Let $d_0 \in D_0$ and $u \in T_1$. As $e(D_0 \cup T_1, D_4) = 0$, $\Gamma(u) \cup \Gamma(d_0) \subset V(G) \setminus D_4$ so

$$\begin{aligned}|\Gamma(d_0) \cap \Gamma(u)| &= d(d_0) + d(u) - |\Gamma(d_0) \cup \Gamma(u)| \geqslant 2\delta(G) + |D_4| - |G| \\ &\geqslant 4\delta(G) - 2|G| - |R_4| > 4|G| - 7\delta(G) - |R_4| \geqslant |R| - |R_4| \\ &\geqslant |\Gamma(d_0) \cap R|,\end{aligned}$$

where we have used inequality (3.4), $\delta(G) > 6/11 \cdot |G|$, inequality (3.3), and $e(d_0, R_4) = 0$ respectively for the final four inequalities. In particular, $d_0$ and $u$ have a common neighbour $d_u \in \Gamma(d_0) \cap D$. □

**Claim 3.41.** *For all $i$ and $d \in D_i$, the sets $\Gamma(d) \cap T_{i-1}$ and $\Gamma(d) \cap T_{i+1}$ are independent.*

*Proof.* Suppose there is $d_0 \in D_0$ such that $\Gamma(d_0) \cap T_1$ contains an edge $uv$. By Claim 3.40, $d_0$ and $u$ have a common neighbour $d_u \in D$. As $d_u$ is adjacent to $u \in R_1$, it must also be adjacent to $v_1$. Similarly there is $d_v \in D$ adjacent to $d_0, v, v_1$. But then $v_1 d_u u v d_v$ is an odd circuit in $G_{d_0}$. □

**Claim 3.42.** *For all $i$, $T_i$ is independent.*

*Proof.* Suppose that $uv$ is an edge in $T_0$. We already have that $D_0$ is independent and $e(D_0, R_0) = 0$ so $u, v \in R_0$. We may choose the edge $uv$ in $R_0$ so that $e(\{u, v, v_1, v_6\})$ is maximal. Apply Lemma 3.37 to $\{u, v, v_1, v_6\}$.

Suppose some $x \in R$ is adjacent to all of $u, v, v_1, v_6$. Then $x \in R_0$. By the maximality of $e(\{u, v, v_1, v_6\})$, we must have had $u$ and $v$ adjacent to both $v_1$, $v_6$ and so $uvv_1v_6$ is a $K_4$.

Hence, there is some $d \in D$ with three neighbours amongst $u$, $v$, $v_1$, $v_6$. Either $d$ is adjacent to both $u$, $v$ or to both $v_1$, $v_6$. If $d$ is adjacent to both $v_1$, $v_6$, then $d \in D_0$. But then $d$ is adjacent to neither $u$ nor $v$, as $e(D_0, R_0) = 0$. Hence $d$ is adjacent to both $u$, $v$. By Claim 3.41, $d \notin D_1 \cup D_6$ so $d \in D_2 \cup D_5$. By symmetry, we may assume that $d = d_2 \in D_2$.

Apply Lemma 3.37 to $\{u, v, d_2, v_5\}$: $d_2uv$ is a triangle so there is $d' \in D$ adjacent to $v_5$ and to two of $u, v, d_2$. As $d'$ is adjacent to $v_5$ and at least one of $u, v \in R_0$, $d'$ is in $D_6$. But then $d'$ is not adjacent to $d_2$ and so is to both $u$ and $v$. However, edge $uv$ lies in $\Gamma(d') \cap R_0$, contrary to Claim 3.41. □

This shows that $G$ is homomorphic to $K_7$ and so is 7-colourable. Before proceeding it will help to give structure to $G_d$ for each $d \in D$.

**Claim 3.43.** *For any $i \in \{0, 1, 2, \ldots, 6\}$ and any $d \in D_i$, $G_d$ is connected bipartite. Further-*

*more, there is a bipartition $G_d$ into independent sets $A_d$ and $B_d$ which satisfy:*

- $(T_{i-1} \cup D_{i+2}) \cap \Gamma(d) \subset A_d$,
- $(T_{i+1} \cup D_{i-2}) \cap \Gamma(d) \subset B_d$,
- *and at least one of* $R_{i+2} \cap \Gamma(d) \subset A_d$, $R_{i-2} \cap \Gamma(d) \subset B_d$ *occurs.*

*Proof.* We may assume $i = 0$. Fix $d \in D_0$ and define for $j = 5, 6, 1, 2$,

$$\begin{aligned} D_j^d &= D_j \cap \Gamma(d), \\ R_j^d &= R_j \cap \Gamma(d), \\ T_j^d &= T_j \cap \Gamma(d) = D_j^d \cup R_j^d, \end{aligned}$$

and note that the $T_j^d$ partition $V(G_d)$. Vertex $v_6 \in G_d$. We let

$$\begin{aligned} A_d &= \{x \in G_d \colon \operatorname{dist}_{G_d}(x, v_6) \text{ is even}\}, \\ B_d &= \{x \in G_d \colon \operatorname{dist}_{G_d}(x, v_6) \text{ is odd}\}. \end{aligned}$$

$G$ is locally bipartite, so $G_d$ is bipartite and so $A_d$ and $B_d$ are independent sets. Now

- $v_6 \in A_d$, $v_1 \in B_d$.
- $v_6$ is adjacent to all of $D_5^d \cup D_1^d$, so $D_5^d \cup D_1^d \subset B_d$.
- $v_1$ is adjacent to all of $D_6^d \cup D_2^d$, so $D_6^d \cup D_2^d \subset A_d$.

We next show that $R_6^d \subset A_d$. Let $x \in R_6^d$: by Claim 3.40, $d$ and $x$ have a common neighbour $d' \in D$. As $d \in D_0$ and $x \in R_6$, $d'$ must be in $D_1 \cup D_5$. Hence, $d' \in B_d$ and so $x \in A_d$. Similarly $R_1^d \subset B_d$.

We now show that at least one of $R_2^d \subset A_d$, $R_5^d \subset B_d$ occurs. If not, then there is $u \in R_2^d \setminus A_d$ and $v \in R_5^d \setminus B_d$. Focus on $u$: $u \notin A_d$ so $\Gamma_{G_d}(u) \subset V(G_d) - B_d \subset T_6^d \cup T_2^d \cup R_5^d$. But $u \in R_2$, the set $T_2$ is independent and $e(R_2, D_6) = 0$, so, in fact,

$$\Gamma_{G_d}(u) \subset R_5^d \cup R_6^d.$$

Similarly,

$$\Gamma_{G_d}(v) \subset R_1^d \cup R_2^d.$$

In particular,

$$\left|\Gamma_{G_d}(u)\right| + \left|\Gamma_{G_d}(v)\right| \leqslant |R_1^d| + |R_2^d| + |R_5^d| + |R_6^d| \leqslant |R|.$$

But then, by inequality (3.3),

$$4|G| - 7\delta(G) \geqslant |R| \geqslant d(d, u) + d(d, v) \geqslant 4\delta(G) - 2|G|,$$

which contradicts $\delta(G) > 6/11 \cdot |G|$.

Finally we need to show that $G_d$ is connected. We will do this by showing $A_d \cup B_d = V(G_d)$. We already have $T_1^d \cup T_6^d \cup D_2^d \cup D_5^d \subset A_d \cup B_d$ and at least one of $R_2^d$, $R_5^d$ is a subset of $A_d \cup B_d$ – we need only show that the other one is too. By symmetry, we may assume $R_2^d \subset A_d \cup B_d$. Fix $x \in R_5^d$. Now $\deg_{G_d}(x) = d(x,d) \geqslant 2\delta(G) - |G| > 0$, so $x$ has some neighbour in $G_d$. But $R_5$ is an independent set, so $x$ has a neighbour in $A_d \cup B_d$ and so $x \in A_d \cup B_d$. □

We are finally in a position to show that $G$ is homomorphic to $\overline{C}_7$. It is here that we will make use of the flexibility in the choice of the $R_i$.

**Claim 3.44.** *It is possible to choose the $R_j$ so that the sets $T_i \cup T_{i+3}$ are all independent.*

*Proof.* Note that $T_i$, $T_{i+3}$, $D_i \cup D_{i+3}$ are all independent and $e(D_i, R_{i+3}) = e(D_{i+3}, R_i) = 0$ so it suffices to show that it is possible to ensure $e(R_i, R_{i+3}) = 0$ for all $i$. We choose the $R_i$ so that

$$S = \sum_{i=0}^{6} e(R_i, R_{i+3})$$

is minimal. Suppose that $S$ is not zero: by symmetry, we may assume there is some $u \in R_2$, $v \in R_6$ with $u$ adjacent to $v$. Apply Lemma 3.37 to $\{u, v, v_0, v_1\}$. Note that any common neighbour of $v_0$, $v_1$ is in $T_2 \cup T_6$ so is adjacent to at most one of $u$, $v$. Moreover, any common neighbour of $v_0$, $v_1$ which lies in $D$ is in $D_2 \cup D_6$ so is adjacent to neither $u$ nor $v$. Hence there is $d \in D$ which is adjacent to both $u$, $v$ and to one of $v_0$, $v_1$. By symmetry, we may assume $d$ is adjacent to $v_1$ and so $d \in D_0$. That is, there is at least one $d \in D_0$ adjacent to both $u$ and $v$.

For any $d \in D_0 \cap \Gamma(u,v)$, consider the bipartition of $G_d$ given by the previous claim:

- $(T_6 \cup D_2) \cap \Gamma(d) \subset A_d$.
- $(T_1 \cup D_5) \cap \Gamma(d) \subset B_d$.
- At least one of $R_2 \cap \Gamma(d) \subset A_d$, $R_5 \cap \Gamma(d) \subset B_d$ occurs.

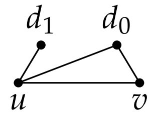
Now $v \in R_6$ is adjacent to $d$, so $v \in A_d$. Also $u$ is adjacent to $v$ and $d$, so $u \in B_d$. But $u \in R_2 \cap \Gamma(d)$ so $R_5 \cap \Gamma(d) \subset B_d$ occurs. Now $\Gamma_{G_d}(u) \subset A_d \subset (T_6 \cup T_2) \cap \Gamma(d)$. But $u \in R_2$, the set $T_2$ is independent and $e(R_2, D_6) = 0$, so

$$\Gamma_{G_d}(u) \subset R_6 \cap \Gamma(d). \tag{3.5}$$

Note that this holds for any choice of $d \in D_0 \cap \Gamma(u,v)$.

We first deal with the case where $u$ has some neighbour in $D_1$. Pick any $d_0 \in \Gamma(u,v) \cap D_0$,

$d_1 \in \Gamma(u) \cap D_1$ and apply Lemma 3.37 to $\{u, v, d_0, d_1\}$.




Vertices $d_0uv$ form a triangle so some $d \in D$ is adjacent to $d_1$ and to two of $d_0$, $u$, $v$. If $d$ is adjacent to $d_0$, then $d \in D_2 \cup D_6$, so $d$ is adjacent to neither $u$ nor $v$. Hence $d \in \Gamma(u, v, d_1) \cap D$, so $d \in D_0$. Thus $d \in \Gamma(u, v) \cap D_0$, $d_1 \in \Gamma(u) \cap D_1$, and $d$ is adjacent to $d_1$. But then $\Gamma_{G_d}(u)$ contains $d_1 \notin R_6$ contradicting (3.5).

We are finally left with the case where $u$ has no neighbours in $D_1$. This means we could have put $u$ in $R_5$ rather than $R_2$ when we chose the $R_i$. In particular, by the minimality of $S$,

$$e(u, R_5) + e(u, R_6) \leqslant e(u, R_1) + e(u, R_2),$$

hence,

$$2\big(e(u, R_5) + e(u, R_6)\big) \leqslant e(u, R_1 \cup R_2 \cup R_5 \cup R_6) \leqslant |R|.$$

Pick any $d \in \Gamma(u, v) \cap D_0$: as $\Gamma_{G_d}(u) \subset R_6 \cap \Gamma(d)$, we have

$$\deg_{G_d}(u) \leqslant e(u, R_6) \leqslant 1/2 \cdot |R|.$$

Thus

$$|R| \geqslant 2d(d, u) \geqslant 4\delta(G) - 2|G| > 4|G| - 7\delta(G) \geqslant |R|,$$

where we used $\delta(G) > 6/11 \cdot |G|$ and inequality (3.3) for the final two inequalities. $\square$

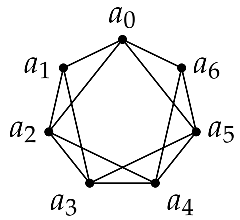
*Proof of Theorem 3.11.* The map which sends all vertices in $T_i$ to vertex $v_i$ is a homomorphism from $G$ to a copy of $\overline{C}_7$. $\square$

### 3.8.2 Proof of Theorem 3.12

In this subsection we prove Theorem 3.12. This has many similarities with the proof of Theorem 3.11, although not having the full symmetry of $\overline{C}_7$ available adds some technicalities.

Fix a locally bipartite graph $G$ with $\delta(G) > 6/11 \cdot |G|$. By Theorem 3.9 and Theorem 3.10, $G$ is either 3-colourable, contains $\overline{C}_7$ or contains $H_2^+$. In the first two cases we are done (using Theorem 3.11), so we assume that $G$ does not contain a copy of $\overline{C}_7$ but does contain a copy of $H_2^+$ (and so also a copy of $H_2$).

We say '$a_0a_1 \ldots a_6$ is a copy of $H_2$ in $G$' to mean that the following configuration appears in $G$. We will continue to use the fact that no vertex is adjacent to five of the vertices which form a copy of $H_2$. We will always consider indices modulo seven.




Our first few claims will nail down to which $a_i$ other vertices may be adjacent. This will eventually allow us to define the sets $D_i$ in a similar way to the proof of Theorem 3.11.

**Claim 3.45.** *Let $a_0a_1 \dots a_6$ be a copy of $H_2$ in $G$. Then there is a vertex $u \notin \{a_0, \dots, a_6\}$ adjacent to $a_5$, $a_0$ and $a_2$ (that is, any copy of $H_2$ 'extends' to a copy of $H_2^+$). Furthermore $|\Gamma(a_5, a_0, a_2)| \geqslant 11\delta(G) - 6|G|$.*

*Proof.* The first part follows immediately from the proof of Claim 3.17. In the language of that claim, all vertices $v$ have $f(v) \leqslant 7$ and any vertex with $f(v) = 7$ either is in a copy of $\overline{C}_7$ or is in $\Gamma(a_5, a_0, a_2)$. As $G$ does not contain $\overline{C}_7$, $\Gamma(a_5, a_0, a_2)$ is exactly the set of vertices with $f(v) = 7$ and all other vertices have $f(v) \leqslant 6$. In particular,

$$11\delta(G) \leqslant \sum_{v \in G} f(v) \leqslant 7|\Gamma(a_5, a_0, a_2)| + 6[|G| - |\Gamma(a_5, a_0, a_2)|] = |\Gamma(a_5, a_0, a_2)| + 6|G|.$$

□

**Claim 3.46.** *Let $a_0a_1 \dots a_6$ be a copy of $H_2$ in $G$. Then no vertex is adjacent to all of $a_6$, $a_0$, $a_1$, $a_3$.*

*Proof.* Suppose some vertex $a$ is adjacent to all of $a_6$, $a_0$, $a_1$, $a_3$. All vertices are adjacent to at most four of the $a_i$, so $a$ cannot be one of the $a_i$. Let $A = \{a, a_0, a_1, \dots, a_6\}$ and gives weights, $\omega$, to the vertices of $A$ as shown.

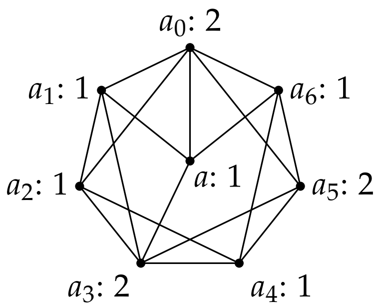



Figure 3.6

For a vertex $v$, let $f(v)$ be the total weight of the neighbours of $v$ in $A$. Now,

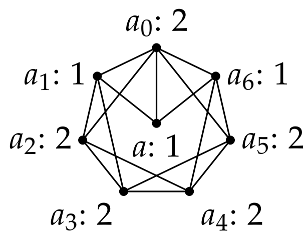
$$\sum_{v \in V(G)} f(v) = \omega(A, G) \geqslant 11\delta(G) > 6|G|,$$

so some vertex $v$ has $f(v) \geqslant 7$. Vertex $v$ is not adjacent to all of $a_0$, $a_3$, $a_5$ else $va_0a_6a_4a_3$ is an odd circuit in $G_{a_5}$ and $v$ is not adjacent to all of $a$, $a_1$, $a_2$, $a_4$, $a_6$ as these form a 5-cycle. Thus $v$ is adjacent to exactly two of $a_0$, $a_3$, $a_5$ and at least three of $a$, $a_1$, $a_2$, $a_4$, $a_6$. As $v$ is adjacent to at most four of the $a_i$, $v$ must be adjacent to $a$.

- If $v$ is adjacent to $a_0$, then $v$ is not adjacent to $a_1$ (else $vaa_0a_1$ is a $K_4$), and $v$ is not adjacent to $a_6$ (else $vaa_6a_0$ is a $K_4$). Thus $v$ is adjacent to $a_2$ and $a_4$. But then $va_0a_1a_3a_4$ is an odd circuit in $G_{a_2}$.
- If $v$ is adjacent to both $a_3$, $a_5$, then $v$ is not adjacent to $a_4$ (else $va_3a_4a_5$ is a $K_4$), and $v$ is not adjacent to $a_1$ (else $vaa_1a_3$ is a $K_4$). Thus $v$ is adjacent to $a_2$ and $a_6$. But then $G_a$ contains the odd circuit $va_6a_0a_1a_3$. □

**Claim 3.47.** *Let $a_0a_1 \ldots a_6$ be a copy of $H_2$ in $G$. Then no vertex is adjacent to all of $a_6$, $a_0$, $a_1$.*

*Proof.* Suppose some vertex $a$ is adjacent to all of $a_6$, $a_0$, $a_1$. All vertices are adjacent to at most four of the $a_i$ so $a$ cannot be one of the $a_i$. Let $A = \{a, a_0, a_1, \ldots, a_6\}$ and give weights, $\omega$, to the vertices of $A$ as shown.




For a vertex $v$, let $f(v)$ be the total weight of the neighbours of $v$ in $A$. Now,

$$\sum_{v \in V(G)} f(v) = \omega(A, G) \geqslant 13\delta(G) > 7|G|,$$

so some vertex $v$ has $f(v) \geqslant 8$. Vertex $v$ must be adjacent to at least three of $a_0$, $a_2$, $a_3$, $a_4$, $a_5$. First suppose that $v$ is adjacent to at least four of $a_0$, $a_2$, $a_3$, $a_4$, $a_5$. Vertex $v$ cannot be adjacent to all of $a_2$, $a_3$, $a_4$ (else $va_2a_3a_4$ is a $K_4$) so $v$ is adjacent to $a_0$ and $a_5$. Similarly $v$ is adjacent to $a_2$. By symmetry, we may assume that $v$ is adjacent to $a_3$. But then $va_0a_1a_4a_3$ is an odd circuit in $G_{a_5}$. Thus $v$ is adjacent to exactly three of $a_0$, $a_2$, $a_3$, $a_4$, $a_5$ and so at least two of $a$, $a_1$, $a_6$. As $v$ is adjacent to at most four of the $a_i$, $v$ must be adjacent to $a$.

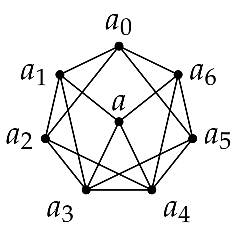
- If $v$ is adjacent to $a_0$, then $v$ cannot be adjacent to $a_1$ (else $vaa_0a_1$ is a $K_4$) and $v$

cannot be adjacent to $a_6$ (else $vaa_6a_0$ is a $K_4$). Thus $v$ is adjacent to only one of $a$, $a_1$, $a_6$ – a contradiction.

- Otherwise $v$ is not adjacent to $a_0$. Certainly $v$ is not adjacent to all of $a_2$, $a_3$, $a_4$ (else $va_2a_3a_4$ is a $K_4$) so $v$ must be adjacent to $a_5$. Similarly $v$ must be adjacent to $a_2$. By symmetry, we may assume that $v$ is adjacent to $a_4$. Then $v$ is not adjacent to $a_6$ (else $va_4a_5a_6$ is a $K_4$) so $v$ is adjacent to $a$ and $a_1$. Thus $v$ is adjacent to $a_1$, $a_2, a_4, a_5$ and $a$. In particular, $v$ is not $a$ nor any of the $a_i$ except possibly $a_3$. But then $a_0a_1a_2va_4a_5a_6$ is a copy of $H_2$ in $G$ and $a$ is adjacent to all of $a_6$, $a_0$, $a_1, v$ which contradicts Claim 3.46. □

**Claim 3.48.** *Let $a_0a_1 \ldots a_6$ be a copy of $H_2$ in $G$. Then no vertex is adjacent to all of $a_1$, $a_3$, $a_4$, $a_6$.*

*Proof.* Suppose vertex $a$ is adjacent to all of $a_1$, $a_3$, $a_4$, $a_6$. All vertices are adjacent to at most four of the $a_i$, so $a$ cannot be an $a_i$. Let $Z = \{a, a_6, a_0, a_1\}$.




We claim that each vertex has at most two neighbours in $Z$. By Claim 3.47, no vertex is adjacent to all of $a_6, a_0, a_1$. If a vertex $v$ is adjacent to all of $a, a_0, a_1$, then $va_0a_2a_3a$ is an odd circuit in $G_{a_1}$ while if $v$ is adjacent to all of $a, a_6, a_0$, then $va_0a_5a_4a$ is an odd circuit in $G_{a_6}$. Finally if $v$ is adjacent to all of $a$, $a_6$, $a_1$, then $va_1a_3a_4a_6$ is an odd circuit in $G_a$. Thus

$$4\delta(G) \leqslant e(Z, G) \leqslant 2|G|,$$

which contradicts $\delta(G) > 6/11 \cdot |G|$. □

We can now show that any vertex with four neighbours in a copy of $H_2$ 'looks like' one of the vertices of the $H_2$.

**Claim 3.49.** *Let $a_0a_1 \ldots a_6$ be a copy of $H_2$ in $G$. Suppose a vertex $v$ is adjacent to at least four of the $a_i$. Then there is $i \in \{0, 2, 3, 4, 5\}$ such that*

$$\Gamma(v) \cap \{a_0, a_1, \ldots, a_6\} = \Gamma(a_i) \cap \{a_0, a_1, \ldots, a_6\}.$$

*Proof.* Fix a vertex $v$ which is adjacent to at least four of the $a_i$. Firstly there is no $i$

with $v$ adjacent to all of $a_{i-1}$, $a_i$, $a_{i+1}$ else there is a $K_4$, or $v$ is adjacent to all of $a_6$, $a_0$, $a_1$ contradicting Claim 3.47. Now suppose there is an $i$ with $v$ adjacent to all of $a_{i-2}$, $a_i$, $a_{i+2}$. If $i = 2, 3, 4, 5$, then $va_{i-2}a_{i-1}a_{i+1}a_{i+2}$ is an odd circuit in $G_{a_i}$. If $i$ is 1 or 6, then, by symmetry, we may assume $i = 1$: vertex $v$ is adjacent to all of $a_6$, $a_1$, $a_3$. Now $v$ is adjacent to none of $a_0$ (by Claim 3.47), $a_2$ (else there is a $K_4$) or $a_4$ (by Claim 3.48). Thus $v$ is adjacent to $a_5$. But then $v$ is adjacent to $a_1$, $a_3$, $a_5$ which is the already discounted case of $i = 3$. Finally if $i = 0$, then $v$ is adjacent to all of $a_5$, $a_0$, $a_2$ so $v$ is adjacent to neither $a_1$ nor $a_6$ (else there is a $K_4$). By symmetry, we may assume that $v$ is adjacent to $a_3$. But then $v$ is adjacent to all of $a_3$, $a_5$, $a_0$ which is the already discounted case of $i = 5$.

Hence there is an $i$ with $v$ adjacent to all of $a_{i-2}$, $a_{i-1}$, $a_{i+1}$, $a_{i+2}$ and no other $a_j$. Now $i$ is not 1 as otherwise $a_0va_2a_3a_4a_5a_6$ is a copy of $\overline{C}_7$. Similarly $i$ is not 6. For all other $i$, $\Gamma(v) \cap \{a_0, a_1, \ldots, a_6\} = \Gamma(a_i) \cap \{a_0, a_1, \ldots, a_6\}$. □

Fix some copy, $v_0v_1 \ldots v_6$, of $H_2$ in $G$. We are ready to build some structure around this copy of $H_2$. Let

$$\begin{aligned} D_i &= \Gamma(v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}) \quad \text{for } i = 0, 2, 3, 4, 5, \\ D_1 &= \Gamma(v_0, v_2, v_3), \qquad\qquad\quad D_6 = \Gamma(v_4, v_5, v_0), \\ D &= \cup_{i=0}^{6} D_i, \qquad\qquad\qquad\quad\; R = V(G) \setminus D. \end{aligned}$$

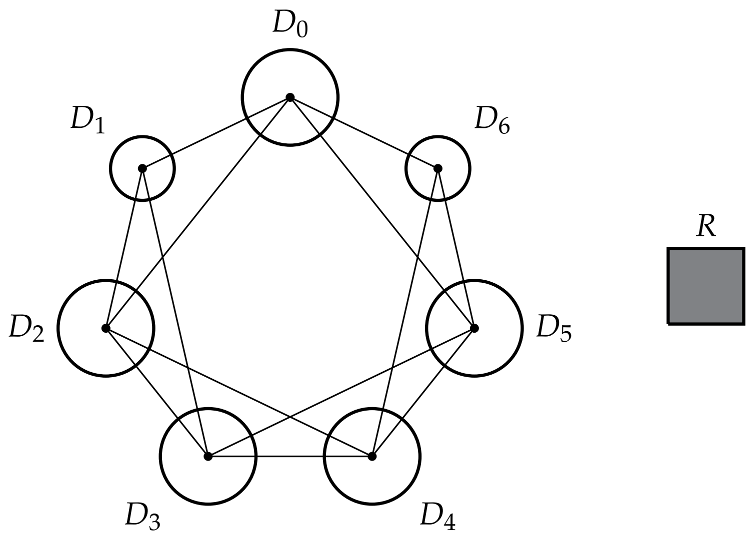
By Claim 3.49, no vertex is adjacent to five of the $v_i$, and so the $D_i$ are pairwise disjoint. Also from Claim 3.49, the vertices adjacent to exactly four of the $v_i$ are those in

$$D^* := D_0 \cup D_2 \cup D_3 \cup D_4 \cup D_5,$$

and all other vertices are adjacent to at most three of the $v_i$. Thus we can give a simple upper bound on the size of $R \cup D_1 \cup D_6 = V(G) \setminus D^*$.

$$\begin{aligned} 7\delta(G) \leqslant e(\{v_0, v_1, \ldots, v_6\}, G) &\leqslant 4|D^*| + 3|R \cup D_1 \cup D_6| = 4|G| - |R \cup D_1 \cup D_6|, \\ \Rightarrow |R \cup D_1 \cup D_6| &\leqslant 4|G| - 7\delta(G). \end{aligned} \tag{3.6}$$

If some $D_i$ has an edge $dd'$, then at least one of $dd'v_{i-2}v_{i-1}$, $dd'v_{i+1}v_{i+2}$ is a $K_4$. Hence each $D_i$ is independent. Suppose there is an edge $dd'$ between $D_i$ and $D_{i+3}$. If $i \neq 5, 6$, then $dv_{i+1}v_{i+2}d'$ is a $K_4$. If $i = 5$, then $d'v_3v_4v_6v_0$ is a 5-cycle in $G_d$ and if $i = 6$, then $dv_0v_1v_3v_4$ is a 5-cycle in $G_{d'}$. Hence $D_i \cup D_{i+3}$ is an independent set for all $i$. Finally, if $d_1 \in D_1$ and $d_6 \in D_6$ are adjacent, then $v_0d_1v_2v_3v_4v_5d_6$ is a copy of $\overline{C}_7$. Thus $G[D]$ is homomorphic to $H_2$. Our aim is to get a handle on $R$.




The following lemma corresponds to Lemma 3.37 and is just as useful.

**Lemma 3.50.** *Let $X \subset V(G)$ be a set of four vertices. Either there is $x \in R \cup D_1 \cup D_6$ adjacent to all of $X$ or there is $x \in D^*$ with at least three neighbours in $X$.*

*Proof.* Using inequality (3.6) and $\delta(G) > 6/11 \cdot |G|$, we have

$$e(X, G) \geqslant 4\delta(G) > 6|G| - 7\delta(G) \geqslant 2|G| + |R \cup D_1 \cup D_6| = 2|D^*| + 3|R \cup D_1 \cup D_6|.$$

But $D^*$, $R \cup D_1 \cup D_6$ partition $V(G)$ so either some vertex in $D^*$ has more than two neighbours in $X$ or some vertex in $R \cup D_1 \cup D_6$ has more than three neighbours in $X$. □

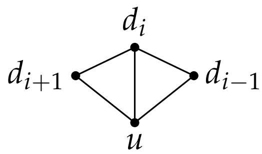
Our first three claims show that the collections of $v_i$ to which vertices can be adjacent are similar to the collections of the $D_i$ in which vertices can have neighbours. This will eventually allow us to define $R_i$ in a similar way to the proof of Theorem 3.11.

**Claim 3.51.** *For all $i$, no vertex has a neighbour in each of $D_{i-1}$, $D_i$, $D_{i+1}$.*

*Proof.* If not, choose $d_{i-1}, d_i, d_{i+1}$ in $D_{i-1}, D_i, D_{i+1}$ respectively with common neighbour $u$. Make the choice of $u$, $d_{i-1}$, $d_i$, $d_{i+1}$ so that $e(\{d_{i-1}, d_i\}) + e(\{d_i, d_{i+1}\})$ is maximal. Apply Lemma 3.50 to $\{u, d_{i-1}, d_i, d_{i+1}\}$.

Suppose some $x$ is adjacent to all of $u$, $d_{i-1}$, $d_i$, $d_{i+1}$. Now apply Lemma 3.50 to $\{u, x, d_{i-1}, d_{i+1}\}$: as $uxd_{i-1}$ and $uxd_{i+1}$ are triangles, there is some $d \in D^*$ adjacent to $d_{i-1}$, $d_{i+1}$ and to one of $u$, $x$. But then $d \in D_i$ so, in our choice of $d_{i-1}$, $d_i$, $d_{i+1}$, $u$ at the start, we could swap $d$ for $d_i$ and swap $u$ for whichever of $u$ and $x$ is adjacent to $d$. This contradicts the maximality of $e(\{d_{i-1}, d_i\}) + e(\{d_i, d_{i+1}\})$ unless $d_i$ is adjacent to both $d_{i-1}$ and $d_{i+1}$. But then $d_i d_{i+1} u x$ is a $K_4$.

Hence, there is some $d \in D^*$ adjacent to three of $u, d_{i-1}, d_i, d_{i+1}$. No vertex in $D$ is adjacent to all of $d_{i-1}, d_i, d_{i+1}$, so $d$ is adjacent to $u$. First suppose that $d$ is adjacent to both $d_{i-1}$ and $d_{i+1}$. Then $d \in D_i$, so, by the maximality at the start, $d_i$ is adjacent to both $d_{i-1}$ and $d_{i+1}$.




- If $i = 0$, then $d_0 d_1 v_2 v_3 v_4 v_5 d_6$ form a copy of $H_2$, but $u$ is adjacent to $d_6 d_0 d_1$, contrary to Claim 3.47.
- If $i \neq 0$, then $v_{i-1}$ is adjacent to $d_{i+1}, v_{i+1}$ while $v_{i+1}$ is adjacent to $d_{i-1}, v_{i-1}$ so $u d_{i-1} v_{i+1} v_{i-1} d_{i+1}$ is an odd circuit in $G_{d_i}$.

Thus $d$ is adjacent to $d_i$. By symmetry, we may assume $d$ is adjacent to $d_{i-1}$. Now $d$ is adjacent to both $d_{i-1}$ and $d_i$, so $d$ is adjacent to both $v_{i-1}$ and $v_i$. But then $u d_{i-1} v_i v_{i-1} d_i$ is an odd circuit in $G_d$. ☐

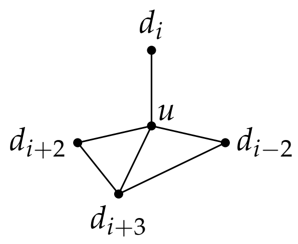
**Claim 3.52.** *For all $i \neq 0$, no vertex has a neighbour in each of $D_{i-2}$, $D_i$, $D_{i+2}$.*

*Proof.* If not, let $i \neq 0$ and choose $d_{i-2}$, $d_i$, $d_{i+2}$ in $D_{i-2}$, $D_i$, $D_{i+2}$ respectively with common neighbour $u$ such that $e(\{d_{i-2}, d_i\}) + e(\{d_i, d_{i+2}\})$ is maximal. No vertex in $D$ has neighbours in each of $D_{i-2}$, $D_i$, $D_{i+2}$, so $u \in R$. Apply Lemma 3.50 to $\{u, d_{i-2}, d_i, d_{i+2}\}$.

Suppose some $x$ is adjacent to all of $u$, $d_{i-2}$, $d_i$, $d_{i+2}$. Now apply Lemma 3.50 to $\{u, x, d_i, d_{i+2}\}$: as $u x d_i$ and $u x d_{i+2}$ are triangles, some $d \in D^*$ is adjacent to $d_i$, $d_{i+2}$ and to one of $u$, $x$. But then $d \in D_{i+1}$ so one of $u, x$ has a neighbour in each of $D_i$, $D_{i+1}$, $D_{i+2}$ contradicting Claim 3.51.

Hence, there is some $d \in D^*$ adjacent to three of $u, d_{i-2}, d_i, d_{i+2}$. No vertex in $D$ is adjacent to all of $d_{i-2}, d_i, d_{i+2}$ so $d$ is adjacent to $u$. First suppose that $d$ is adjacent to $d_i$. By symmetry we may assume that $d$ is adjacent to $d_{i+2}$. But then $d \in D_{i+1}$ and so $u$ has a neighbour in each of $D_i$, $D_{i+1}$ and $D_{i+2}$, contrary to Claim 3.51. Hence $d$ is adjacent to both $d_{i-2}, d_{i+2}$ so $d \in D_i \cup D_{i-3} \cup D_{i+3}$.

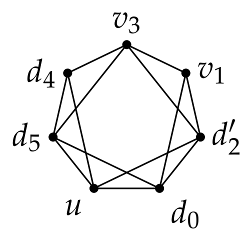
Suppose $d \in D_i$: $i \neq 0$ so $v_{i-1}, v_{i+1}$ are adjacent so $u d_{i-2} v_{i-1} v_{i+1} d_{i+2}$ is an odd circuit in $G_d$. Thus $d \in D_{i-3} \cup D_{i+3}$. By symmetry we may assume that $d \in D_{i+3}$. Write $d_{i+3}$ for $d$.




We will show that there is $d_i' \in D_i$ adjacent to both $u$ and $d_{i-2}$. Apply Lemma 3.50 to $\{u, d_i, v_{i-1}, d_{i-2}\}$: by Claim 3.51, no vertex is adjacent to all of $d_i$, $v_{i-1}$, $d_{i-2}$ so there is $d' \in D^*$ adjacent to $u$ and two of $d_i$, $v_{i-1}$, $d_{i-2}$.

- If $d'$ is adjacent to $d_i$ and $d_{i-2}$, then $d' \in D_{i-1}$ so $u$ has a neighbour in each of $D_{i-2}$, $D_{i-1}$, $D_i$, contrary to Claim 3.51.
- If $d'$ is adjacent to $v_{i-1}$ and $d_i$, then $d' \in D_{i-2} \cup D_{i+1}$. But if $d' \in D_{i+1}$, then $u$ has a neighbour in each of $D_i$, $D_{i+1}$, $D_{i+2}$, so $d' \in D_{i-2}$. By the maximality at the start, we must have $d_{i-2}$ adjacent to $d_i$. We may take $d_i' = d_i$.
- If $d'$ is adjacent to $v_{i-1}$ and $d_{i-2}$, then $d' \in D_i \cup D_{i-3}$. If $d' \in D_{i-3}$, then $u$ has a neighbour in each of $D_{i+2}$, $D_{i+3}$ and $D_{i-3}$, contrary to Claim 3.51, so $d' \in D_i$. We may take $d_i' = d'$

Thus there is some $d_i' \in D_i$ adjacent to both $u$ and $d_{i-2}$. If $i \neq 2$, then $v_{i-1}$ and $v_{i-3}$ are adjacent, so $d_i' u d_{i+3} v_{i-3} v_{i-1}$ is an odd circuit in $G_{d_{i-2}}$. Finally if $i = 2$, then $v_3 d_4 d_5 u d_0 d_2' v_1$ is a copy of $H_2$ in $G$ (note that all the vertices are distinct: vertex $u \in R$ and the others are in distinct $D_i$). However, $v_2$ is adjacent to all of $v_1$, $v_3$, $d_4$ contrary to Claim 3.47.



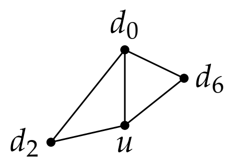

□

**Claim 3.53.** *No vertex has a neighbour in each of $D_6$, $D_0$, $D_2$ and no vertex has a neighbour in each of $D_5$, $D_0$, $D_1$.*

*Proof.* Suppose not – by symmetry we may assume that some vertex has a neighbour in each of $D_6$, $D_0$, $D_2$. Choose $d_6 \in D_6$, $d_0 \in D_0$ and $d_2 \in D_2$ with common neighbour $u$ such that $e(\{d_6, d_0\}) + e(\{d_0, d_2\})$ is maximal. We first show that $d_0$ is adjacent to both $d_2$ and $d_6$. Apply Lemma 3.50 to $\{u, d_6, d_0, d_2\}$.

Suppose some $x$ is adjacent to all of $u$, $d_6$, $d_0$, $d_2$. Now apply Lemma 3.50 to $\{u, x, d_0, d_2\}$: $uxd_0$ and $uxd_2$ are triangles so some vertex in $D^*$ is adjacent to $d_0$ and $d_2$ and one of $u$, $x$. But no vertex in $D^*$ has a neighbour in each of $D_0$, $D_2$.

Hence, there is some $d \in D^*$ adjacent to three of $u$, $d_6$, $d_0$, $d_2$. No vertex in $D^*$ has a neighbour in each of $D_0$, $D_2$ so $d$ is adjacent to $u$, $d_6$ and one of $d_0$, $d_2$. If $d$ is adjacent to $d_0$, then $d \in D_5$. But then $u$ has a neighbour in each of $D_5$, $D_6$ and $D_0$, contrary to Claim 3.51. Thus $d$ is adjacent to $d_2$ and so $d \in D_0 \cup D_4$. If $d \in D_4$, then $u$ has a neighbour in each of $D_2$, $D_4$ and $D_6$, contradicting Claim 3.52. Hence $d \in D_0$. By the maximality at the start, we must have $d_0$ adjacent to both $d_2$ and $d_6$.




Now $d_0v_1d_2v_3v_4v_5d_6$ is a copy of $H_2$ so can be extended, by Claim 3.45, to a copy of $H_2^+$. That is, there is some other vertex, $v$, adjacent to $v_5$, $d_0$, $d_2$. But then $G_{d_0}$ contains the odd circuit $ud_2vv_5d_6$. □

**Corollary 3.54.** *Every $v$ in $G$ satisfies one of the following properties.*

- *$v$ has a neighbour in each of $D_5$, $D_0$, $D_2$ and $\Gamma(v) \cap D \subset D_5 \cup D_0 \cup D_2$.*
- *There is an $i$ such that $\Gamma(v) \cap D \subset \Gamma(v_i) \cap D$.*

*Proof.* Fix a vertex $v$. First suppose that $v$ has a neighbour in each of $D_5$, $D_0$, $D_2$. By Claim 3.51, $v$ has no neighbours in $D_1 \cup D_6$. By Claim 3.52, $v$ has no neighbours in $D_3 \cup D_4$. Thus $\Gamma(v) \cap D \subset D_5 \cup D_0 \cup D_2$.

Otherwise $v$ does not have a neighbour in each of $D_5$, $D_0$, $D_2$. By Claims 3.51 and 3.52, there is an $i$ with $\Gamma(v) \cap D \subset D_{i-2} \cup D_{i-1} \cup D_{i+1} \cup D_{i+2}$. If $i \neq 1, 6$, then $D_{i-2} \cup D_{i-1} \cup D_{i+1} \cup D_{i+2} = \Gamma(v_i) \cap D$ and so we are done. Otherwise we may assume, by symmetry, that $i = 1$: $\Gamma(v) \cap D \subset D_6 \cup D_0 \cup D_2 \cup D_3$. By Claim 3.53, we have one of the following.

- $\Gamma(v) \cap D \subset D_0 \cup D_2 \cup D_3 = \Gamma(v_1) \cap D$,
- $\Gamma(v) \cap D \subset D_6 \cup D_2 \cup D_3 \subset \Gamma(v_4) \cap D$,
- $\Gamma(v) \cap D \subset D_6 \cup D_0 \cup D_3 \subset \Gamma(v_5) \cap D$. □

This corollary gives structure to $R$. Firstly let

$$R_{502} = \{v \in R \colon v \text{ has a neighbour in each of } D_5, D_0, D_2\}.$$

Then, for $i = 0, 1, \ldots, 6$ choose

$$R_i \subset \{v \in R \colon \Gamma(v) \cap D \subset \Gamma(v_i) \cup D\},$$

so that $R_{502} \cup R_0 \cup R_1 \cup \cdots \cup R_6$ is a partition of $R$. There may be some flexibility in the choice of the $R_i$ – we will make use of this later. For now we just take any arbitrary choice. Note, for each $i$, that

$$e(R_i, D_i \cup D_{i-3} \cup D_{i+3}) = 0,$$

and also that

$$e(R_1, D_6) = e(R_6, D_1) = e(R_{502}, D_1 \cup D_3 \cup D_4 \cup D_6) = 0.$$

For $i = 0, 1, \ldots, 6$, let $T_i = D_i \cup R_i$. We may give a lower bound for the size of $T_i$. Firstly,

$$\begin{aligned} d(v_1) + d(v_6) &= |D_0| + |D| - |D_1| - |D_6| + |R \cap (\Gamma(v_1) \cup \Gamma(v_6))| + |R \cap \Gamma(v_1, v_6)| \\ &\leqslant |D_0| + |D| - |D_1| - |D_6| + |R| - |R_1| - |R_6| - |R_{502}| + |R_0|, \end{aligned}$$

so

$$|T_0| \geqslant 2\delta(G) - |G| + |T_1| + |T_6| + |R_{502}|. \tag{3.7}$$

Next,

$$\begin{aligned} d(v_0) + d(v_2) &= |D_1| + |D| + |R \cap (\Gamma(v_0) \cup \Gamma(v_2))| + |R \cap \Gamma(v_0, v_2)| \\ &\leqslant |D_1| + |D| + |R| + |R_1| + |R_{502}|, \end{aligned}$$

and so (using symmetry for the second inequality)

$$\begin{aligned} |T_1| + |R_{502}| &\geqslant 2\delta(G) - |G|, \\ |T_6| + |R_{502}| &\geqslant 2\delta(G) - |G|. \end{aligned} \tag{3.8}$$

A similar argument applied to $d(v_1) + d(v_3)$ gives

$$\begin{aligned} |T_2| &\geqslant 2\delta(G) - |G| + |T_6| + |R_{502}|, \\ |T_5| &\geqslant 2\delta(G) - |G| + |T_1| + |R_{502}|, \end{aligned} \tag{3.9}$$

and one applied to $d(v_2) + d(v_4)$ gives

$$\begin{aligned} |T_3| &\geqslant 2\delta(G) - |G|, \\ |T_4| &\geqslant 2\delta(G) - |G|. \end{aligned} \tag{3.10}$$

In particular, for all $i \neq 1, 6$,

$$|T_i| \geqslant 2\delta(G) - |G|, \tag{3.11}$$

and, using inequality (3.8) combined with (3.7) and (3.9), we have for $i = 0, 2, 5$,

$$|T_i| \geqslant 4\delta(G) - 2|G|. \tag{3.12}$$

The next three claims are technical in nature but will speed up what follows.

**Claim 3.55.** *Every $d \in D_i$ and every $u \in R_{i+1}$ have a common neighbour in $D^*$ unless $i = 2$, in which case they have a common neighbour in $D^* \cup R_{502}$. Similarly, every $d \in D_i$ and every $u \in R_{i-1}$ have a common neighbour in $D^*$ unless $i = 5$, in which case they have a common neighbour in $D^* \cup R_{502}$.*

*Proof.* It is enough to prove the assertion when $d \in D_i$ and $u \in R_{i+1}$, the other assertion following symmetrically. Now $\Gamma(d) \cup \Gamma(u) \subset V(G) \setminus D_{i-3}$, so

$$|\Gamma(d) \cap \Gamma(u)| = d(d) + d(u) - |\Gamma(d) \cup \Gamma(u)| \geqslant 2\delta(G) - |G| + |D_{i-3}|.$$

Now, for $i = 0, 1, 3, 5, 6$, inequality (3.11) gives

$$\begin{aligned} |\Gamma(d) \cap \Gamma(u)| &\geqslant 2\delta(G) - |G| + |D_{i-3}| \geqslant 4\delta(G) - 2|G| - |R_{i-3}| \\ &> 4|G| - 7\delta(G) - |R_{i-3}| \geqslant |R \cup D_1 \cup D_6| - |R_{i-3}| \\ &\geqslant |\Gamma(d) \cap (R \cup D_1 \cup D_6)|, \end{aligned}$$

where we used $\delta(G) > 6/11 \cdot |G|$, inequality (3.6) and $e(D_i, R_{i-3}) = 0$ in the third, fourth and fifth inequalities respectively. Hence there is a common neighbour of $d$ and $u$ which is not in $R \cup D_1 \cup D_6$, so is in $D^*$.

For $i = 4$, inequality (3.8) gives

$$\begin{aligned} |\Gamma(d) \cap \Gamma(u)| &\geqslant 2\delta(G) - |G| + |D_1| \geqslant 4\delta(G) - 2|G| - |R_1| - |R_{502}| \\ &> 4|G| - 7\delta(G) - |R_1| - |R_{502}| \geqslant |R \cup D_1 \cup D_6| - |R_1| - |R_{502}| \\ &\geqslant |\Gamma(d) \cap (R \cup D_1 \cup D_6)|, \end{aligned}$$

where we used $\delta(G) > 6/11 \cdot |G|$, inequality (3.6) and $e(D_4, R_1 \cup R_{502}) = 0$ in the third, fourth and fifth inequalities respectively.

For $i = 2$, inequality (3.8) gives

$$\begin{aligned}|\Gamma(d) \cap \Gamma(u)| &\geqslant 2\delta(G) - |G| + |D_6| \geqslant 4\delta(G) - 2|G| - |R_6| - |R_{502}| \\ &> |R \cup D_1 \cup D_6| - |R_6| - |R_{502}| \geqslant |\Gamma(d) \cap ((R \setminus R_{502}) \cup D_1 \cup D_6)|. \square\end{aligned}$$

**Claim 3.56.** *Every $d \in D_0$ and every $u \in R_{502}$ have a common neighbour in $D_2 \cup D_5$. Every $d \in D_2 \cup D_5$ and every $u \in R_{502}$ have a common neighbour in $D_0$.*

*Proof.* Fix $d \in D_0$ and $u \in R_{502}$. Now $\Gamma(d) \cup \Gamma(u) \subset V(G) \setminus D_3$, so, as in the previous claim,

$$\begin{aligned}|\Gamma(d) \cap \Gamma(u)| &\geqslant 4\delta(G) - 2|G| - |R_3| > |R \cup D_1 \cup D_6| - |R_3| \\ &\geqslant |\Gamma(d) \cap (R \cup D_1 \cup D_6)|,\end{aligned}$$

so $d, u$ have a common neighbour in $D^*$. But $d \in D_0$ so this common neighbour must be in $D_2 \cup D_5$.

Suppose, for contradiction, $d \in D_2$, $u \in R_{502}$ have no common neighbour in $D_0$: then $d, u$ have no common neighbour in $D$. Now $u \in R_{502}$ so $u$ has a neighbour $d_0 \in D_0$, $d_5 \in D_5$. Apply Lemma 3.50 to $\{u, d, d_0, d_5\}$: no vertex in $D^*$ is adjacent to $d$, $d_0$ or to $d_0$, $d_5$ or to $d$, $u$ so there is some $v$ adjacent to all of $u$, $d$, $d_0$, $d_5$.

Now apply Lemma 3.50 to $\{u, v, d_0, d_5\}$: as $uvd_0$ and $uvd_5$ are triangles, some vertex in $D^*$ is adjacent to both $d_0$, $d_5$ (and one of $u$, $v$). But no vertex in $D^*$ has a neighbour in each of $D_0$, $D_5$. $\square$

**Claim 3.57.** *For each $i \in \{0, 1, \dots, 6, 502\}$, every two vertices in $R_i$ have a common neighbour in $D^*$.*

*Proof.* We first deal with $i \in \{0, 1, \dots, 6\}$. If $u, v \in R_i$ have no common neighbour in $D^*$, then $\Gamma(u) \cap D^*$ and $\Gamma(v) \cap D^*$ are disjoint subsets of $D_{i-2} \cup D_{i-1} \cup D_{i+1} \cup D_{i+2}$. Now, by inequality (3.6),

$$|\Gamma(u) \cap D^*| \geqslant d(u) - |R \cup D_1 \cup D_6| \geqslant 8\delta(G) - 4|G|,$$

so $|D_{i-2} \cup D_{i-1} \cup D_{i+1} \cup D_{i+2}| \geqslant 16\delta(G) - 8|G|$. But then, using inequalities (3.7) to (3.10) [note that at most one of $i, i-3, i+3$ can be 1 or 6],

$$\begin{aligned}|G| &\geqslant |D_{i-2} \cup D_{i-1} \cup D_{i+1} \cup D_{i+2}| + |R_{502}| + |T_i| + |T_{i+3}| + |T_{i-3}| \\ &\geqslant 16\delta(G) - 8|G| + 3(2\delta(G) - |G|) = 22\delta(G) - 11|G|,\end{aligned}$$

which contradicts $\delta(G) > 6/11 \cdot |G|$.

Now we deal with $i = 502$. If $u, v \in R_{502}$ have no common neighbour in $D^*$, then, as above,

$$|D_5 \cup D_0 \cup D_2| \geqslant 16\delta(G) - 8|G|.$$

But then, using inequalities (3.8) and (3.10),

$$\begin{aligned} |G| &\geqslant |D_5 \cup D_0 \cup D_2| + |R_{502}| + |T_1| + |T_3| + |T_4| \\ &\geqslant 16\delta(G) - 8|G| + 3(2\delta(G) - |G|) = 22\delta(G) - 11|G|, \end{aligned}$$

which contradicts $\delta(G) > 6/11 \cdot |G|$. □

We now aim to show that $T_i$ is independent for all $i$.

**Claim 3.58.** *For all $i$ and $d \in D_i$: $\Gamma(d) \cap T_{i-1}$ and $\Gamma(d) \cap T_{i+1}$ are independent.*

*Proof.* Suppose there is $d \in D_i$ such that $\Gamma(d) \cap T_{i+1}$ contains the edge $uv$. As $D_{i+1}$ is independent and $e(D_{i+1}, R_{i+1}) = 0$, both $u, v \in R_{i+1}$.

We first deal with the case when $i$ is not 2. Then, by Claim 3.55, there is $d_u \in D^*$ adjacent to both $u, d_i$. As $d_u$ is adjacent to $u \in R_{i+1}$, $d_u$ is adjacent to $v_{i+1}$. Similarly there is $d_v \in D^*$ adjacent to $v$, $d_i$, $v_{i+1}$. But then $v_{i+1}d_u uvd_v$ is an odd circuit in $G_{d_i}$.

Now suppose $i = 2$ and write $d = d_2$. By Claim 3.55, $d_2$ and $u$ have a common neighbour $x_u \in D^* \cup R_{502}$. If $x_u \in D^*$, then as $x_u$ is adjacent to $u \in R_3$, $x_u$ is adjacent to $v_3$, while if $x_u \in R_{502}$, then, by Claim 3.56, $x_u$ and $d_2$ have a common neighbour $d' \in D_0$. Taking $d_u = v_3$ in the former case and $d_u = d'$ in the latter, we see that $x_u$ and $d_2$ have a common neighbour $d_u \in D$ which is adjacent to $v_1$. Similarly $d_2$ and $v$ have a common neighbour $x_v$ such that $x_v$ and $d_2$ have a common neighbour $d_v$ which is adjacent to $v_1$. But then $v_1 d_u x_u uvx_v d_v$ is an odd circuit in $G_{d_2}$. □

**Claim 3.59.** *For all $i$, $T_i$ is independent.*

*Proof.* Suppose $uv$ is an edge in $T_i$ – as $D_i$ is independent and $e(D_i, R_i) = 0$ we have $u, v \in R_i$. By Claim 3.57, $u, v$ have a common neighbour in $d \in D^*$. By Claim 3.58, $d \notin D_{i-1} \cup D_{i+1}$ so $d \in D_{i-2} \cup D_{i+2}$. We complete the argument assuming that $d \in D_{i-2}$. The other case is analogous. Write $d = d_{i-2}$.

Apply Lemma 3.50 to $\{u, v, d_{i-2}, v_{i+2}\}$: as $uvd_{i-2}$ is a triangle, there is $d' \in D^*$ adjacent to $v_{i+2}$ and to two of $u$, $v$, $d_{i-2}$. In particular, $d'$ is adjacent to $v_{i+2}$ and to at least one of $u, v \in R_i$, so $d' \in D_{i+1}$. But then $d'$ cannot be adjacent to $d_{i-2}$ and so is adjacent to both

$u$ and $v$. However, edge $uv$ lies in $\Gamma(d') \cap R_i$, contradicting Claim 3.58. $\square$

**Claim 3.60.** *$R_{502}$ is independent.*

*Proof.* Suppose $uv$ is an edge in $R_{502}$. By Claim 3.57, $u, v$ have a common neighbour $d \in D_5 \cup D_0 \cup D_2$.

First suppose that $d \in D_0$. By Claim 3.56, $d$, $u$ have a common neighbour $d_u \in D_2 \cup D_5$ and $d$, $v$ have a common neighbour $d_v \in D_2 \cup D_5$. We may assume that $d_u \in D_2$.

- If $d_v \in D_2$, then $v_1 d_u uv d_v$ is an odd circuit in $G_d$.
- If $d_v \in D_5$, then $dv_1 d_u v_3 v_4 d_v v_6$ is a copy of $H_2$ so, by Claim 3.45, can be extended to a copy of $H_2^+$: there is $x$ adjacent to all of $d$, $d_u$, $d_v$. But then $xd_u uv d_v$ is an odd circuit in $G_d$.

Next suppose that $d \in D_2 \cup D_5$. By symmetry, we may assume that $d \in D_2$. By Claim 3.56, $d, u$ have a common neighbour $d_u \in D_0$ and $d, v$ have a common neighbour $d_v \in D_0$. But then $v_1 d_u uv d_v$ is an odd circuit in $G_d$. $\square$

We have made good progress: we now know that $G$ is homomorphic to $K_8$, so is 8-colourable.

**Claim 3.61.** *$e(R_{502}, R_1 \cup R_6) = 0$.*

*Proof.* If not, then we may assume there is an edge $uv$ with $u \in R_{502}$ and $v \in R_1$. We first show that $u$, $v$ have a common neighbour in $D_0 \cup D_2$. Apply Lemma 3.50 to $\{u, v, v_0, v_2\}$: any common neighbour of $v_0, v_2$ is in $T_1 \cup R_{502}$ so is adjacent to at most one of $u, v$. Hence there is $d \in D^*$ adjacent to three of $u$, $v$, $v_0$, $v_2$. No vertex of $D^*$ is adjacent to both $v_0$, $v_2$ so $d$ is a common neighbour of $u$, $v$. As $d$ is adjacent to $u$ and $v$, $d \in D_0 \cup D_2$.

First suppose $d \in D_0$. By Claim 3.55, $d$, $v$ have a common neighbour $d_v \in D^*$: $d_v$ is adjacent to both $d$, $v$ so $d_v \in D_2$. By Claim 3.56, $d$, $u$ have a common neighbour $d_u \in D_2 \cup D_5$.

- If $d_u \in D_2$, then $v_1 d_u uv d_v$ is an odd circuit in $G_d$.
- If $d_u \in D_5$, then $dv_1 d_v v_3 v_4 d_u v_6$ is a copy of $H_2$ so, by Claim 3.45, there is a vertex $x$ adjacent to all of $d$, $d_u$, $d_v$. But then $xd_u uv d_v$ is an odd circuit in $G_d$.

Now suppose $d \in D_2$. By Claim 3.56, $d, u$ have a common neighbour $d_u \in D_0$. By Claim 3.55, $d, v$ have a common neighbour $d_v \in D^*$: $d_v$ is adjacent to both $d$, $v$ so $d_v \in D_0 \cup D_3$. But then $v_1 d_u uv d_v$ is an odd circuit in $G_d$. $\square$

**Claim 3.62.** $e(R_1, R_6) = 0$.

*Proof.* If not, then there is an edge $uv$ with $u \in R_1$, $v \in R_6$. We first show that $u$, $v$ have a common neighbour $d \in D_0$. Apply Lemma 3.50 to $\{v_0, v_2, u, v\}$: any common neighbour of $v_0, v_2$ is in $T_1 \cup R_{502}$ so is not adjacent to $u$. Hence there is $d \in D^*$ adjacent to three of $v_0, v_2, u, v$. No vertex in $D^*$ is adjacent to both $v_0$ and $v_2$, so $d$ is adjacent to both $u$ and $v$. But $u \in R_1$, $v \in R_6$, so $d \in D_0$.

By Claim 3.55, $d$, $u$ have a common neighbour $d_u \in D^*$ and $d$, $v$ have a common neighbour $d_v \in D^*$. As $d \in D_0$ and $u \in R_1$, we have $d_u \in D_2$. Similarly, $d_v \in D_5$. Now $dv_1d_uv_3v_4d_vv_6$ is a copy of $H_2$, so, by Claim 3.45, there is a vertex $x$ adjacent to all of $d$, $d_u$, $d_v$. But then $xd_uuvd_v$ is an odd circuit in $G_d$. □

Before proceeding it will help to give structure to $G_d$ for each $d \in D_1 \cup D_2 \cup \cdots \cup D_6$. This corresponds to Claim 3.43 in the proof of Theorem 3.11.

**Claim 3.63.** *For each $i \in \{1, 2, \ldots, 6\}$ and every $d \in D_i$, $G_d$ is connected bipartite. Furthermore, there is a bipartition of $G_d$ into independent sets $A_d$ and $B_d$ which satisfy :*

- $(T_{i-1} \cup D_{i+2}) \cap \Gamma(d) \subset A_d$,
- $(T_{i+1} \cup D_{i-2}) \cap \Gamma(d) \subset B_d$,
- *and at least one of* $R_{i+2} \cap \Gamma(d) \subset A_d$, $R_{i-2} \cap \Gamma(d) \subset B_d$ *occurs.*

*If $i = 2$, then $R_{502} \cap \Gamma(d) \subset A_d$ and if $i = 5$, then $R_{502} \cap \Gamma(d) \subset B_d$.*

*Proof.* Fix $d \in D_i$ and define for $j = i-2, i-1, i+1, i+2$,

$$\begin{aligned} D_j^d &= D_j \cap \Gamma(d), \\ R_j^d &= R_j \cap \Gamma(d), \\ R_{502}^d &= R_{502} \cap \Gamma(d), \end{aligned}$$

and note that these partition $V(G_d)$ (and some of them can be empty). Also let $T_j^d = T_j \cap \Gamma(d)$. Vertex $v_{i-1} \in G_d$. We let

$$\begin{aligned} A_d &= \{x \in G_d \colon \operatorname{dist}_{G_d}(x, v_{i-1}) \text{ is even}\}, \\ B_d &= \{x \in G_d \colon \operatorname{dist}_{G_d}(x, v_{i-1}) \text{ is odd}\}. \end{aligned}$$

$G$ is locally bipartite so $G_d$ is bipartite and so $A_d$ and $B_d$ are independent sets. Now, as $i$ is not 0,

- $v_{i-1} \in A_d$, $v_{i+1} \in B_d$.
- $v_{i-1}$ is adjacent to all of $D_{i-2}^d \cup D_{i+1}^d$, so $D_{i-2}^d \cup D_{i+1}^d \subset B_d$.

- $v_{i+1}$ is adjacent to all of $D^d_{i-1} \cup D^d_{i+2}$, so $D^d_{i-1} \cup D^d_{i+2} \subset A_d$.

If $i = 2$, then, by Claim 3.56, any $x \in R^d_{502}$ has a neighbour in $D^d_0 \subset B_d$, so $R^d_{502} \subset A_d$. If $i = 5$, then, by Claim 3.56, any $x \in R^d_{502}$ has a neighbour in $D^d_0 \subset A_d$ so $R^d_{502} \subset B_d$. For other $i$, $R^d_{502}$ is empty.

We next show that $R^d_{i-1} \subset A_d$ and $R^d_{i+1} \subset B_d$. Fix $x \in R^d_{i-1}$ – it suffices to show $x \in A_d$. Suppose $x$ and $d$ have a common neighbour in $D$. As $x \in R_{i-1}$, $e(x, D_{i-1}) = e(x, D_{i+2}) = 0$, so $x$ has a neighbour in $D^d_{i-2} \cup D^d_{i+1} \subset B_d$, so $x \in A_d$. On the other hand if $x$ and $d$ do not have a common neighbour in $D$, then Claim 3.55 guarantees that $i = 5$ and $x$ has a neighbour in $R^d_{502} \subset B_d$, so $x \in A_d$. Similarly $R^d_{i+1} \subset B_d$.

We now show that at least one of $R^d_{i+2} \subset A_d$, $R^d_{i-2} \subset B_d$ occurs. If not, then there is $u \in R^d_{i+2} \setminus A_d$ and $v \in R^d_{i-2} \setminus B_d$. Focus on $u$: $u \notin A_d$ so $\Gamma_{G_d}(u) \subset V(G_d) - B_d \subset T^d_{i-1} \cup T^d_{i+2} \cup R^d_{i-2} \cup R^d_{502}$. But $u \in R_{i+2}$, the set $T_{i+2}$ is independent and $e(R_{i+2}, D_{i-1}) = 0$, so, in fact,

$$\Gamma_{G_d}(u) \subset R^d_{i-1} \cup R^d_{i-2} \cup R^d_{502}.$$

Similarly

$$\Gamma_{G_d}(v) \subset R^d_{i+1} \cup R^d_{i+2} \cup R^d_{502}.$$

If $u$ and $v$ both have a neighbour in $R^d_{502}$, then $R^d_{502}$ would be non-empty, so $i = 2, 5$ and either $R^d_{502} \subset A_d$ or $R^d_{502} \subset B_d$. The former contradicts $v \notin B_d$ and the latter contradicts $u \notin A_d$. Thus, at most one of $u$ and $v$ has a neighbour in $R^d_{502}$. In particular,

$$\left|\Gamma_{G_d}(u)\right| + \left|\Gamma_{G_d}(v)\right| \leqslant |R^d_{i-2}| + |R^d_{i-1}| + |R^d_{i+1}| + |R^d_{i+2}| + |R^d_{502}| \leqslant |R|.$$

But then inequality (3.6) gives

$$4|G| - 7\delta(G) \geqslant |R| \geqslant d(d,u) + d(d,v) \geqslant 4\delta(G) - 2|G|,$$

which contradicts $\delta(G) > 6/11 \cdot |G|$.

Finally we need to show that $G_d$ is connected. We will do this by showing $A_d \cup B_d = V(G_d)$. We already have $T^d_{i-1} \cup T^d_{i+1} \cup D^d_{i-2} \cup D^d_{i+2} \cup R^d_{502} \subset A_d \cup B_d$ and at least one of $R^d_{i-2}$, $R^d_{i+2}$ is a subset of $A_d \cup B_d$ – we need only show that the other one is too. By symmetry, we may assume $R^d_{i+2} \subset A_d \cup B_d$. Fix $x \in R^d_{i-2}$. Now $\deg_{G_d}(x) = d(x,d) \geqslant 2\delta(G) - |G| > 0$, so $x$ has some neighbour in $G_d$. But $R_{i-2}$ is an independent set, so $x$ has a neighbour in $A_d \cup B_d$ and so $x \in A_d \cup B_d$. $\square$

To prove that $G$ is homomorphic to $H_2^+$ we would need to show that $e(R_i, R_{i+3}) = 0$ for all $i$ and $e(R_3 \cup R_4, R_{502}) = 0$. We make a start.

**Claim 3.64.** *$e(R_i, R_{i+3}) = 0$ for $i = 0, 2, 4$.*

*Proof.* Suppose not: there is an edge $uv$ with $u \in R_{i+3}$, $v \in R_i$. We first show that $u$ and $v$ have a common neighbour $d \in D^* \cap (D_{i+1} \cup D_{i+2})$. Apply Lemma 3.50 to $\{u, v, v_{i+1}, v_{i+2}\}$: any common neighbour of $v_{i+1}, v_{i+2}$ is in $T_i \cup T_{i+3}$ so is adjacent to at most one of $u$ and $v$. Moreover, any common neighbour of $v_{i+1}$ and $v_{i+2}$ in $D^*$ is in $D_i \cup D_{i+3}$ and so is adjacent to neither $u$ nor $v$. Hence there is $d \in D^*$ adjacent to both $u$, $v$ and one of $v_{i+1}, v_{i+2}$ – in particular, $d \in D_{i+1} \cup D_{i+2}$.

If $i = 2$, we may take $d \in D_3$, by symmetry. If $i = 0, 4$ we may assume, by symmetry that $i = 4$. Since $d \in D^*$, we have $d \in D_5$. In conclusion, we have adjacent vertices $u \in R_{i+3}$, $v \in R_i$ with common neighbour $d \in D_{i+1}$ where $i$ is 2 or 4. Consider the bipartition of $G_d$ given by Claim 3.63:

- $(T_i \cup D_{i+3}) \cap \Gamma(d) \subset A_d$.
- $(T_{i+2} \cup D_{i-1} \cup R_{502}) \cap \Gamma(d) \subset B_d$.
- At least one of $R_{i+3} \cap \Gamma(d) \subset A_d$ or $R_{i-1} \cap \Gamma(d) \subset B_d$ occurs.

As $v \in R_i$, we have $v \in A_d$ and so $u \in B_d$. But $u \in R_{i+3} \cap \Gamma(d)$, so $R_{i-1} \cap \Gamma(d) \subset B_d$ occurs. Now $\Gamma_{G_d}(u) \subset A_d \subset (T_i \cup T_{i+3}) \cap \Gamma(d)$. But $u \in R_{i+3}$, the set $T_{i+3}$ is independent and $e(R_{i+3}, D_i) = 0$, so

$$\Gamma_{G_d}(u) \subset R_i \cap \Gamma(d).$$

Thus, $|R_i| \geqslant d(d, u) \geqslant 2\delta(G) - |G|$. But then, using inequalities (3.6) and (3.8),

$$4|G| - 7\delta(G) \geqslant |R \cup D_1 \cup D_6| \geqslant |R_i| + |T_1 \cup R_{502}| \geqslant 4\delta(G) - 2|G|,$$

which contradicts $\delta(G) > 6/11 \cdot |G|$. □

We have been flexible about the $R_i$ and so all of our results thus far hold for any $R_i$ satisfying their definition. Now is the time to make a further choice. We choose the $R_i$ so that

$$S = \sum_{i=0}^{6} e(R_i, R_{i+3}) + e(R_3 \cup R_4, R_{502}) \tag{3.13}$$

is minimal.

**Claim 3.65.** *$e(R_i, R_{i+3}) = 0$ for $i = 1, 3$.*

*Proof.* By symmetry it suffices to prove this for $i = 3$. Suppose we have $u \in R_6$, $v \in R_3$ with $u$ adjacent to $v$. We first show that $u$ and $v$ have a common neighbour $d \in D_4 \cup D_5$. Apply Lemma 3.50 to $\{u, v, v_4, v_5\}$: any common neighbour of $v_4$ and $v_5$ is in $T_3 \cup T_6$, so is adjacent to at most one of $u$ and $v$. Moreover, any common neighbour of $v_4$ and $v_5$ in $D^*$ is in $D_3 \cup D_6$, so is adjacent to neither $u$ nor $v$. Hence, there is $d \in D^*$ adjacent to both $u$, $v$ and to one of $v_4$, $v_5$ – in particular, $d \in D_4 \cup D_5$. When $d \in D_4$, the argument of Claim 3.64 works again (with $i = 3$). We deal with the more difficult $d \in D_5$ case. For any $d \in D_5 \cap \Gamma(u, v)$, consider the bipartition of $G_d$ given by Claim 3.63:

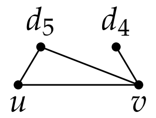
- $(T_4 \cup D_0) \cap \Gamma(d) \subset A_d$.
- $(T_6 \cup D_3 \cup R_{502}) \cap \Gamma(d) \subset B_d$.
- At least one of $R_0 \cap \Gamma(d) \subset A_d$ or $R_3 \cap \Gamma(d) \subset B_d$ occurs.

As $u \in R_6 \subset B_d$, we must have $v \in A_d$. But $v \in R_3 \cap \Gamma(d)$, so $R_0 \cap \Gamma(d) \subset A_d$ occurs. Now $\Gamma_{G_d}(v) \subset B_d \subset T_6 \cup T_3 \cup R_{502}$. But $v \in R_3$, the set $T_3$ is indepdendent and $e(R_3, D_6) = 0$, so

$$\Gamma_{G_d}(v) \subset (R_6 \cup R_{502}) \cap \Gamma(d).$$

Note that this holds for any choice of $d \in D_5 \cap \Gamma(u, v)$.

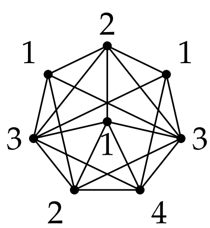
We first deal with the case where $v$ has at least one neighbour in $D_4$. Pick any $d_5 \in \Gamma(u, v) \cap D_5$, $d_4 \in \Gamma(v) \cap D_4$.




Apply Lemma 3.50 to $\{u, v, d_4, d_5\}$: the vertices $d_5$, $u$, $v$ form a triangle, so some $d \in D^*$ is adjacent to $d_4$ and to two of $d_5$, $u$, and $v$. If $d$ is adjacent to $d_5$, then $d \in D_3 \cup D_6$, so $d$ is adjacent to neither $u$ nor $v$. Hence $d \in \Gamma(u, v, d_4) \cap D^*$, so $d \in D_5$. But then $d \in \Gamma(u, v) \cap D_5$ and $\Gamma_{G_d}(v)$ contains $d_4 \notin (R_6 \cup R_{502}) \cap \Gamma(d)$, a contradiction.

We are finally left with the case where $v$ has no neighbours in $D_4$: when we were choosing the $R_i$ we could have put $u$ in $R_0$. In particular, Claim 3.64 gives $e(u, R_4) = 0$. Also $R_3$ is independent, so $e(u, R_3) = 0$. Thus if we put $u \in R_0$, then $u$ would contribute 0 to $S$ while currently it contributes at least 1 (the edge $uv$ contributes to $e(R_6, R_3)$). This contradicts the minimality of $S$. □

*Proof of Theorem 3.12.* Identifying $T_i$ with $v_i$ and $R_{502}$ with a single vertex gives a homomorphism from $G$ to the following graph, which is $H_2^+$ with four extra edges, but which is still 4-colourable (colouring shown in the diagram). □



### 3.8.3 Proof of Theorem 3.13

In this subsection we prove Theorem 3.13. Thankfully we may use all of the machinery from our proof of Theorem 3.12 and, in particular, we only need to show that $e(R_3 \cup R_4, R_{502}) = e(R_1, R_5) = e(R_2, R_6) = 0$. As before, we choose the $R_i$ so that $S$, as given in equation (3.13), is minimal. Using inequalities (3.10) and (3.12), we have

$$|T_3|, |T_4| \geqslant 2\delta(G) - |G| > (1/9 - 2\varepsilon)|G|,$$
$$|T_0|, |T_2|, |T_5| \geqslant 4\delta(G) - 2|G| > (2/9 - 4\varepsilon)|G|.$$

Also, by Claim 3.45, we have

$$|R_{502}| \geqslant |\Gamma(v_5, v_0, v_2)| \geqslant 11\delta(G) - 6|G| > (1/9 - 11\varepsilon)|G|.$$

Now $2(1/9 - 2\varepsilon) + 3(2/9 - 4\varepsilon) + (1/9 - 11\varepsilon) = 1 - 27\varepsilon$, so in fact we have

$$|T_3|, |T_4|, |R_{502}| = (1/9 - \mathcal{O}(\varepsilon))|G|,$$
$$|T_0|, |T_2|, |T_5| = (2/9 - \mathcal{O}(\varepsilon))|G|.$$

Throughout we will use $\mathcal{O}(\varepsilon)$ to denote a quantity for which there is an absolute positive constant $C$ (in particular, independent of $G$ and $\varepsilon$) such that the quantity lies between $-C\varepsilon$ and $C\varepsilon$. By inequality (3.6),

$$|D_1 \cup D_6 \cup R| \leqslant 4|G| - 7\delta(G) < (1/9 + 7\varepsilon)|G|.$$

Putting all this together (and noting that $R_{502} \subset R$) we have

$$\begin{aligned} |R \setminus R_{502}|, |D_1|, |D_6| &= \mathcal{O}(\varepsilon)|G|, \\ |D_3|, |D_4|, |R_{502}| &= (1/9 + \mathcal{O}(\varepsilon))|G|, \\ |D_0|, |D_2|, |D_5| &= (2/9 + \mathcal{O}(\varepsilon))|G|. \end{aligned} \tag{3.14}$$

Note that these numbers match the weighting of $H_2^+$ given in Figure 3.3 on page 33. That was a weighting of $H_2^+$ with minimum degree attaining $5/9$.

**Claim 3.66.** *Provided $\varepsilon > 0$ is sufficiently small, $e(R_1, R_5) = e(R_2, R_6) = 0$.*

*Proof.* Suppose this is false. By symmetry we may take $r_2 \in R_2$ and $r_6 \in R_6$ where $r_2 r_6$ is an edge. We first claim that $r_2$ has a neighbour $t_1 \in T_1$. Indeed, if $r_2$ does not, then, when we chose the $R_i$, we could have put $r_2$ in $R_5$. Thus, by the minimality of $S$,

$$e(r_2, R_5) + e(r_2, R_6) \leqslant e(r_2, R_1) + e(r_2, R_2).$$

However, the left-hand is positive (the edge $r_2r_6$ contributes to it), while the right-hand side is zero ($R_2$ is independent and $r_2$ has no neighbours in $R_1$ by assumption). Thus $r_2$ does indeed have a neighbour in $t_1 \in T_1$.

Now, $\Gamma(r_2) \subset T_0 \cup T_1 \cup T_3 \cup T_4 \cup R_{502} \cup R_6$ and this union has size $(5/9 + \mathcal{O}(\varepsilon))|G|$, so $r_2$ has at most $\mathcal{O}(\varepsilon)|G|$ non-neighbours in $D_0 \cup D_3 \cup D_4$. Also $\Gamma(t_1) \subset T_0 \cup T_2 \cup T_3 \cup R_5$ and this union has size $(5/9 + \mathcal{O}(\varepsilon))|G|$, so $t_1$ has at most $\mathcal{O}(\varepsilon)|G|$ non-neighbours in $D_0 \cup D_3$. Similarly, $r_6$ has at most $\mathcal{O}(\varepsilon)|G|$ non-neighbours in $D_0 \cup D_4$. But $D_0, D_3$ both have size at least $(1/9 + \mathcal{O}(\varepsilon))|G|$, so, provided $\varepsilon$ is small enough, there is $d_0 \in D_0$ adjacent to all of $r_2$, $t_1$, $r_6$ and there is $d_3 \in D_3$ adjacent to both $t_1$, $r_2$.

Finally, $\Gamma(d_3) \subset T_1 \cup T_2 \cup T_4 \cup T_5$ and this union has size $(5/9 + \mathcal{O}(\varepsilon))|G|$, so $d_3$ has at most $\mathcal{O}(\varepsilon)|G|$ non-neighbours in $D_4$. But $D_4$ has size $(1/9 + \mathcal{O}(\varepsilon))|G|$, so, provided $\varepsilon$ is small enough, there is $d_4 \in D_4$ adjacent to all of $r_2$, $d_3$, and $r_6$. But then $d_0t_1d_3d_4r_6$ is a 5-cycle in $G_{r_2}$. □

**Claim 3.67.** *Provided $\varepsilon > 0$ is sufficiently small, $e(R_3 \cup R_4, R_{502}) = 0$.*

*Proof.* Suppose this is false. By symmetry we may take some $r_3 \in R_3$ that has at least one neighbour in $R_{502}$. We first claim that $r_3$ has at least one neighbour in $D_4$. Indeed, if $r_3$ does not, then, when we chose the $R_i$, we could have put $r_3$ in $R_0$. Thus, by the minimality of $S$,

$$e(r_3, R_0) + e(r_3, R_6) + e(r_3, R_{502}) \leqslant e(r_3, R_3) + e(r_3, R_4).$$

But we showed in Claim 3.64 that $e(R_0, R_4) = 0$ and we did this before we made the choice to minimise $S$. In particular, as we could have put $r_3$ in $R_0$ we know that $e(r_3, R_4) = 0$. Also $R_3$ is independent so $e(r_3, R_3) = 0$. But then, the right-hand side of the inequality is zero, while the left-hand side is positive ($r_3$ has at least one neighbour in $R_{502}$).

Thus $r_3$ has at least one neighbour in $R_{502}$ and at least one neighbour in $D_4$. We may write,

$$|\Gamma(r_3) \cap D_4| = c_4|G|, \quad |\Gamma(r_3) \cap R_{502}| = c_{502}|G|,$$

where $0 < c_4, c_{502} \leqslant 1/9 + \mathcal{O}(\varepsilon)$ (using our knowledge of $|D_4|$, $|R_{502}|$). Also,

$$\begin{aligned} |\Gamma(r_3) \cap D_5| &\geqslant d(r_3) - |\Gamma(r_3) \cap R_{502}| - |\Gamma(r_3) \cap D_4| - |T_2| - |T_1| - |R \setminus R_{502}| \\ &\geqslant (1/3 - c_4 - c_{502} + \mathcal{O}(\varepsilon))|G|. \end{aligned}$$

But $1/3 - c_4 - c_{502} \geqslant 1/9 - \mathcal{O}(\varepsilon)$, so, provided $\varepsilon$ is sufficiently small, $r_3$ has at least one neighbour in $D_5$.

We next claim that the configuration in Figure 3.7 appears with $d_5 \in D_5$, $d_4 \in D_4$, $r_{502} \in R_{502}$.

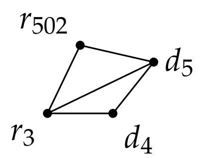



Figure 3.7

First suppose that $c_4, c_{502} \geqslant 1/27$. Pick a neighbour $d_5 \in D_5$ of $r_3$. Now $\Gamma(d_5) \subset T_3 \cup T_4 \cup T_6 \cup T_0 \cup R_{502}$ and this union has size $(5/9 + \mathcal{O}(\varepsilon))|G|$, so $d_5$ has at most $\mathcal{O}(\varepsilon)|G|$ non-neighbours in $D_4 \cup R_{502}$. But $\Gamma(r_3) \cap D_4$ and $\Gamma(r_3) \cap R_{502}$ both have size at least $1/27 \cdot |G|$, so, provided $\varepsilon$ is small enough, $d_5$ has a neighbour in each of $\Gamma(r_3) \cap D_4$, $\Gamma(r_3) \cap R_{502}$ giving the configuration in Figure 3.7.

Otherwise $\min\{c_4, c_{502}\} < 1/27$ and so

$$\begin{aligned} |\Gamma(r_3) \cap D_5| &\geqslant (1/3 - c_4 - c_{502} + \mathcal{O}(\varepsilon))|G| \\ &\geqslant (1/3 - 1/9 - 1/27 + \mathcal{O}(\varepsilon))|G| = (5/27 + \mathcal{O}(\varepsilon))|G|. \end{aligned}$$

Pick $r_{502} \in R_{502}$ and $d_4 \in D_4$ both adjacent to $r_3$. Now, $\Gamma(d_4) \subset T_2 \cup T_3 \cup T_5 \cup T_6$ and this union has size $(5/9 + \mathcal{O}(\varepsilon))|G|$, so $d_4$ has at most $\mathcal{O}(\varepsilon)|G|$ non-neighbours in $D_5$. Also, $\Gamma(r_{502}) \subset T_0 \cup T_2 \cup T_5 \cup R_3 \cup R_4$ and this union has size $(2/3 + \mathcal{O}(\varepsilon))|G|$, so $r_{502}$ has at most $(1/9 + \mathcal{O}(\varepsilon))|G|$ non-neighbours in $D_5$. But $5/27 > 1/9$, so, provided $\varepsilon$ is sufficiently small, there is some $d_5 \in \Gamma(r_3) \cap D_5$ adjacent to both $d_4$ and $r_{502}$.

Hence, in all cases, the configuration in Figure 3.7 appears with $d_5 \in D_5$, $d_4 \in D_4$, and $r_{502} \in R_{502}$. Consider the bipartition of $G_{d_5}$ given by Claim 3.63:

- $(T_4 \cup D_0) \cap \Gamma(d_5) \subset A_{d_5}$.
- $(T_6 \cup D_3 \cup R_{502}) \cap \Gamma(d_5) \subset B_{d_5}$.

In particular, $d_4 \in A_{d_5}$ and $r_{502} \in B_{d_5}$. But then $r_3 \in G_{d_5}$ has a neighbour in both $A_{d_5}$ and $B_{d_5}$, so can be in neither, which is a contradiction. □

*Proof of Theorem 3.13.* Take $\varepsilon > 0$ sufficiently small so that Claims 3.66 and 3.67 both hold. Identifying $T_i$ with $v_i$ and $R_{502}$ with a single vertex gives a homomorphism from $G$ to $H_2^+$. □

# CHAPTER 4

# THE CHROMATIC PROFILE OF LOCALLY COLOURABLE GRAPHS

Many of the results of this chapter have been submitted in a forthcoming paper [Ill21d].

## 4.1 INTRODUCTION

The previous chapter considered the chromatic profile of locally bipartite graphs. Here we consider the chromatic profile and threshold of more general locally colourable graphs. We take the liberty to slightly generalise this notion.

**Definition 4.1.** *A graph is $a$-**locally** $b$-**partite** if the common neighbourhood of every $a$-clique is $b$-colourable. A graph is **locally** $b$-**partite** if it is 1-locally $b$-partite: the neighbourhood of every vertex is $b$-colourable. $\mathcal{F}_{a,b}$ denotes the family of $a$-locally $b$-partite graphs.*

We note in passing that the family of $a$-locally $b$-partite graphs is both monotone and closed under taking blow-ups. Also note that

$$\mathcal{F}_{1,\ell} \subset \mathcal{F}_{2,\ell-1} \subset \cdots \subset \mathcal{F}_{\ell,1} = \{G\colon G \text{ is } K_{\ell+2}\text{-free}\}.$$

In particular, $\mathcal{F}_{a,b}$ is a natural subfamily of $K_{\ell+2}$-free graphs where $\ell = a + b - 1$. As mentioned previously, Goddard and Lyle [GL10], and Nikiforov [Nik10] independently showed that the chromatic profile of $K_{r+2}$-free graphs can be derived straightforwardly from that of triangle-free graphs. Thus the chromatic profile of $K_{\ell+2}$-free graphs, that is, of $\mathcal{F}_{\ell,1}$, has been completely determined.

The chromatic profiles of the family $\mathcal{F}_{a,b}$ for $b \geqslant 2$ and for $b = 1$ display substantially different behaviour, however. Particularly striking is that the chromatic threshold (and so all chromatic profile values) for $K_{a+b+1}$-free graphs are greater than the first interesting value for $a$-locally $b$-partite ones ($b \geqslant 2$).

We determine the chromatic threshold of $\mathcal{F}_{a,b}$ in § 4.2. In the particular case $a = 1$ and $b = 2$ (that is, for locally bipartite graphs), we recover the results of Allen et al. [ABG+13] and Łuczak and Thomassé [ŁT10] that this chromatic threshold is $1/2$. Our construction for the lower bound simplifies Łuczak and Thomassé's by using Schrijver graphs [Sch78] in place of their topological Borsuk-bicap graphs.

**Theorem 4.2** (chromatic thresholds)**.** *Let $a$ and $b$ be positive integers with $b \geqslant 2$ and let $\ell = a + b - 1$. Then the chromatic threshold of the family of $a$-locally $b$-partite graphs is*

$$\delta_\chi(\mathcal{F}_{a,b}) = 1 - \frac{1}{\ell}.$$

*In particular, the chromatic threshold of locally $b$-partite graphs is $1 - 1/b$.*

For comparison, the chromatic threshold of the family $\mathcal{F}_{\ell,1}$ of $K_{\ell+2}$-free graphs is $1 - 1/(\ell + 1/2)$.

Now for the chromatic profile. All $a$-locally $b$-partite graphs are $K_{a+b+1}$-free and, furthermore, the $n$-vertex $K_{a+b+1}$-free graph with highest minimum degree (and most edges), the Turán graph $T_{a+b}(n)$, is $a$-locally $b$-partite and $(a + b)$-chromatic. In particular,

$$\delta_\chi(\mathcal{F}_{a,b}, k) = 1 - \frac{1}{a+b} \text{ for } k = 1, 2, \ldots, \ell := a + b - 1,$$

since there are no $K_{a+b+1}$-free (and so no $a$-locally $b$-partite) graphs with $\delta(G) > (1 - 1/(a + b)) \cdot |G|$. In particular, the first interesting value in the chromatic profile of $a$-locally $b$-partite graphs is $\delta_\chi(\mathcal{F}_{a,b}, \ell + 1)$. For $a$-locally bipartite graphs, we obtain a result corresponding to the $4/7$ result for locally bipartite graphs. Here suitable blow-ups of $K_{a-1} + \overline{C}_7$ give the lower bound.

**Theorem 4.3** ($a$-locally bipartite graphs)**.** *Let $a$ be a positive integer and let $\ell = a + 1$. Then*

$$\delta_\chi(\mathcal{F}_{a,2}, \ell + 1) = 1 - \frac{1}{\ell + 1/3}.$$

We have the following upper bound for $a$-locally $b$-partite graphs (and Theorem 4.2 provides a lower bound).

**Theorem 4.4** ($a$-locally $b$-partite graphs)**.** *Let $a$ and $b$ be positive integers with $b \geqslant 3$ and let*

*$\ell = a + b - 1$. Then*

$$\delta_\chi(\mathcal{F}_{a,b}, \ell + 1) \leqslant 1 - \frac{1}{\ell + 1/7},$$

*and, in particular, every locally $b$-partite graph $G$ with $\delta(G) > (1 - 1/(b + 1/7)) \cdot |G|$ is $(b+1)$-colourable.*

Note that the chromatic threshold of the family of $K_{\ell+2}$-free graphs is $1 - 1/(\ell + 1/2)$, so all the chromatic profile values of $K_{\ell+2}$-free graphs are greater than the first interesting value for $a$-locally $b$-partite ones ($b \geqslant 2$).

To extend the chromatic profile of triangle-free graphs to $K_{r+1}$-free graphs as mentioned above, Goddard and Lyle, and Nikiforov showed that every $n$-vertex maximal $K_{r+1}$-free graph with minimum degree greater than $\delta_\chi(K_{r+1}) \cdot n$ consists of an independent set joined to a $K_r$-free graph. That is, maximal graphs of $\mathcal{F}_{r,1}$ with sufficiently large minimum degree are obtained from those in $\mathcal{F}_{r-1,1}$ by joining an independent set. A simple induction then converts the structure of triangle-free graphs to the structure of $K_{r+1}$-free graphs. It is natural to ask whether something similar can be done to convert between different $\mathcal{F}_{a,b}$. Firstly, there does not seem to be an easy way to convert between $\mathcal{F}_{a,b-1}$ and $\mathcal{F}_{a,b}$ (certainly joining on an independent set fails). Although we obtain the upper bound for $\delta_\chi(\mathcal{F}_{1,b}, b+1)$ in Theorem 4.4 using knowledge of locally bipartite graphs, it is far from a straightforward induction.

Joining on an independent set to a graph in $\mathcal{F}_{a-1,b}$ does give a graph in $\mathcal{F}_{a,b}$, but it is not clear that all maximal graphs in $\mathcal{F}_{a,b}$ of large minimum degree are obtained in this way – the lower value of $\delta_\chi(\mathcal{F}_{a,b})$ for $b \geqslant 2$ means the structural lemma of Goddard, Lyle, and Nikiforov does not apply. While our arguments extending results from locally $b$-partite graphs to $a$-locally $b$-partite graphs are not so complicated, they interestingly do require knowledge of locally $b'$-partite graphs for all $b' \leqslant a + b$. It seems that the crux of understanding locally colourable graphs is understanding locally $b$-partite graphs, a sentiment we will crystallise in § 4.4.

## 4.2 Chromatic Thresholds

The $(r-1)$-locally 1-partite graphs are exactly the $K_{r+1}$-free ones and their chromatic threshold was determined by Goddard and Lyle [GL10], and Nikiforov [Nik10].

$$\delta_\chi(\mathcal{F}_{r-1,1}) = \delta_\chi(K_{r+1}) = 1 - \frac{1}{r - 1/2}.$$

In this section we determine $\delta_\chi(\mathcal{F}_{a,b})$ for all positive $a$ and $b \geqslant 2$ by proving Theorem 4.2. We repeat the theorem for convenience.

**Theorem 4.2** (chromatic thresholds)**.** *Let $a$ and $b$ be positive integers with $b \geqslant 2$ and let $\ell = a + b - 1$. Then the chromatic threshold of the family of $a$-locally $b$-partite graphs is*

$$\delta_\chi(\mathcal{F}_{a,b}) = 1 - \frac{1}{\ell}.$$

*In particular, the chromatic threshold of locally $b$-partite graphs is $1 - 1/b$.*

The upper bound $\delta_\chi(\mathcal{F}_{a,b}) \leqslant 1 - 1/(a+b-1)$ needs a result of Allen et al. [ABG+13].

*Proof of upper bound in Theorem 4.2.* Fix $a$ and $b$ positive integers with $b \geqslant 2$ and let $\ell = a + b - 1$. Let $d > 1 - 1/\ell$ and let $G \in \mathcal{F}_{a,b}$ with $\delta(G) \geqslant d|G|$. Now $K_{\ell-1} + C_5$ has an $a$-clique whose common neighbourhoods contains $K_{b-2} + C_5$ so is not $b$-colourable. In particular $G$ is $(K_{\ell-1} + C_5)$-free.

Now, in the language of [ABG+13], $K_{\ell-1} + C_5$ is $(\ell+2)$-near-acyclic, so the chromatic threshold of the family of $(K_{\ell-1} + C_5)$-free graphs is $1 - 1/\ell$. In particular, there is a constant $C$ depending only upon $d$ and $\ell$ (and not on $G$) such that $\chi(G) \leqslant C$. Hence

$$\delta_\chi(\mathcal{F}_{a,b}) \leqslant 1 - \frac{1}{\ell} = 1 - \frac{1}{a+b-1}. \qquad \square$$

For the lower bound it suffices to give examples of graphs $G \in \mathcal{F}_{a,b}$ with $\delta(G) \geqslant (1 - 1/(a+b-1) - o(1)) \cdot |G|$ which have arbitrarily large chromatic number. Allen et al. [ABG+13] used graphs of large girth and high chromatic number as well as Borsuk-Hajnal graphs for their lower bounds. However, these depend upon the forbidden subgraph $H$ and do not seem to be applicable here where the collection of forbidden subgraphs is infinite. Łuczak and Thomassé [ŁT10] modified a Borsuk-bicap graph to give the lower bound of $1/2$ for the chromatic threshold of locally bipartite graphs. We will give a somewhat simpler example which gives the lower bound for all $\mathcal{F}_{a,b}$ ($a \geqslant 1, b \geqslant 2$).

Our example is based upon the classical *Schrijver graph* [Sch78]. The *Kneser graph*, $\mathrm{KG}(n,k)$, is the graph whose vertex set is all $k$-subsets of $\{1, 2, \dots, n\}$ with two vertices adjacent if the corresponding $k$-sets are disjoint. The question of the chromatic number of this graph was first raised by Kneser [Kne55] in 1955. He conjectured that $\chi(\mathrm{KG}(n,k)) = n - 2k + 2$, which corresponds to the colouring where all sets with smallest element $i$ get colour $i$ ($i = 1, 2, \dots, n - 2k + 1$) and all other sets get colour $n - 2k + 2$. The conjecture was first proved by Lovász in 1978 [Lov78] using homotopy theory and shortly afterwards Bárány [Bár78] gave a beautiful proof that seemed to be from the book. Amazingly, almost twenty five years later, Greene [Gre02], whilst still an undergraduate, further simplified Bárány's already succinct proof!

Soon after the original proofs of Kneser's conjecture, Schrijver [Sch78] found a vertex-critical subgraph of the Kneser graph with a simple description. The *Schrijver graph*, $\mathrm{SG}(n,k)$, is the graph whose vertex set is all $k$-subsets of $\{1,2,\ldots,n\}$ that do not contain both $i$ and $i+1$ for any $i=1,2,\ldots,n-1$ and do not contain both $n$ and 1. Again vertices are adjacent if the corresponding $k$-sets are disjoint. Put another way, $\mathrm{SG}(n,k)$ is the induced subgraph of $\mathrm{KG}(n,k)$ obtained by deleting all vertices whose corresponding sets are supersets of any of $\{1,2\}$, $\{2,3\}$, $\{3,4\}, \ldots, \{n-1,n\}$, $\{n,1\}$. Schrijver showed that $\mathrm{SG}(n,k)$ is vertex-critical with chromatic number $\chi(\mathrm{SG}(n,k)) = \chi(\mathrm{KG}(n,k)) = n-2k+2$. A recent interesting development in the area was a description of an edge-critical subgraph of the Schrijver graph (and so also of the Kneser graph) by Kaiser and Stehlík [KS20a; KS20b].

*Proof of lower bound in Theorem 4.2.* Fix $a$ and $b$ positive integers with $b \geqslant 2$ and let $\ell = a+b-1$. Fix $k$ and let $n = 2k+f(k)$ where $f(k)$ is a non-negative integer less than $k$, and both $f(k) \to \infty$, $f(k)/k \to 0$ as $k \to \infty$. Note that $\mathrm{SG}(n,k)$ is triangle-free: there are not three pairwise-disjoint subsets of $\{1,2,\ldots,n\}$ each of size $k > n/3$. We will eventually consider a blow-up of the graph shown in Figure 4.1.

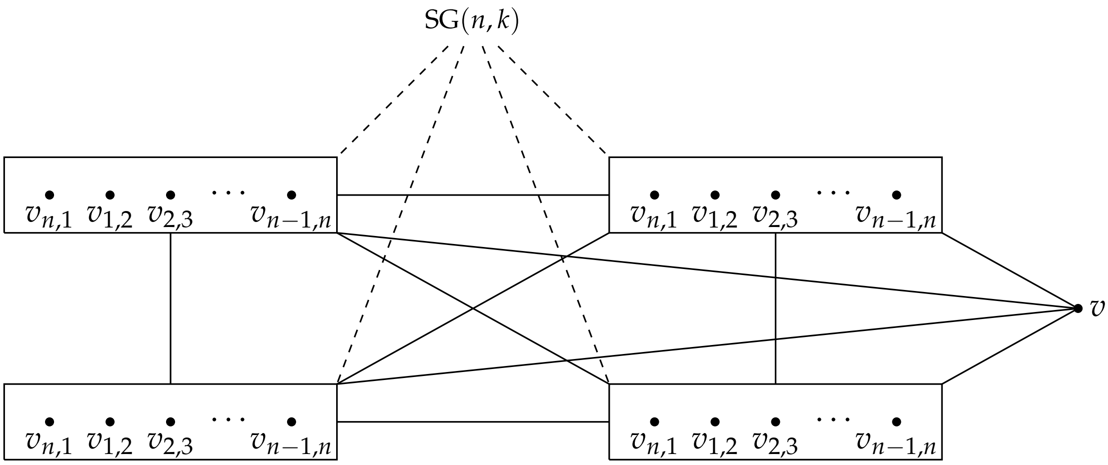



Figure 4.1

- Each rectangle is an independent set of $n$ vertices: $v_{1,2}, v_{2,3}, \ldots, v_{n-1,n}, v_{n,1}$ – we always consider indices modulo $n$.
- There are $\ell-1$ rectangles and the vertices in rectangles form a complete $(\ell-1)$-partite graph with $n$ vertices in each part.
- The vertex $v$ is adjacent to all of the $v_{i,i+1}$ but has no neighbours in the copy of $\mathrm{SG}(n,k)$ – in particular, the rectangles together with $v$ form a complete $\ell$-partite graph.
- Finally, $A \in \mathrm{SG}(n,k)$ is adjacent to $v_{i,i+1}$ if either $i$ or $i+1$ is in $A$ (note, by the definition of the Schrijver graph, that it is impossible for both $i$ and $i+1$ to be in

$A$).

We first check that the graph, $G$, shown in Figure 4.1 is locally $\ell$-partite. Fix a vertex $u$ of $G$. If $u = v$, then $G_u$ is the $\ell - 1$ rectangles and so is $(\ell - 1)$-colourable. Next suppose that $u$ is a vertex in the copy of $\mathrm{SG}(n,k)$, so $v \notin \Gamma(u)$. As $\mathrm{SG}(n,k)$ is triangle-free, $\Gamma(u)$ consists of an independent set in $\mathrm{SG}(n,k)$ together with some vertices from the $\ell - 1$ rectangles. In particular, $G_u$ is $\ell$-colourable.

Finally suppose that $u$ lies in the $\ell - 1$ rectangles. By symmetry, we may take $u$ to be a $v_{1,2}$. Then $\Gamma(u)$ consists of two independent sets in $\mathrm{SG}(n,k)$ (sets containing 1 and sets containing 2), $\ell - 2$ rectangles and $v$. There are no edges from $v$ to the copy of $\mathrm{SG}(n,k)$, so $G_u$ is $\ell$-colourable.

Now we consider a suitable blow-up of $G$. Let $s$ be a multiple of $n$ that is much larger than $|\mathrm{SG}(n,k)|$. We will blow-up each $v_{i,i+1}$ by $s/n$ so that all rectangles contain $s$ vertices and we will blow up $v$ by $s$ also. Note that $v$ together with the rectangles form a complete $\ell$-partite graph with $s$ vertices in each part. Keep the copy of $\mathrm{SG}(n,k)$ as it is. Call the resulting graph $G'$.

- $G'$ has $\ell s + |\mathrm{SG}(n,k)|$ vertices.
- Any $A \in \mathrm{SG}(n,k)$ is adjacent to $2k/n$ proportion of the $v_{i,i+1}$ so has degree at least $(\ell - 1)s(2k)/n = (\ell - 1)s/(1 + f(k)/(2k))$.
- Any other vertex has degree at least $(\ell - 1)s$.
- $\chi(G') \geqslant \chi(\mathrm{SG}(n,k)) = n - 2k + 2 = f(k) + 2$.

Given any $C, \varepsilon > 0$, we may choose $k$ large enough so that $f(k) \geqslant C$ and $(\ell - 1)/(1 + f(k)/(2k)) \geqslant \ell - 1 - \varepsilon$ and then choose $s$ large enough so that $\ell s + |\mathrm{SG}(n,k)| \leqslant (\ell + \varepsilon)s$. The resulting graph $G'$ has chromatic number at least $f(k) + 2 \geqslant C$ and

$$\frac{\delta(G')}{|G'|} \geqslant \frac{\ell - 1 - \varepsilon}{\ell + \varepsilon} \geqslant 1 - \frac{1}{\ell} - \varepsilon.$$

Being $a$-locally $b$-partite is preserved when taking blow-ups. Now $G$ is locally $\ell$-partite and so $G'$ is too. As $\mathcal{F}_{1,\ell} \subset \mathcal{F}_{a,b}$, $G'$ is $a$-locally $b$-partite. Thus,

$$\delta_\chi(\mathcal{F}_{a,b}) \geqslant 1 - \frac{1}{\ell} - \varepsilon = 1 - \frac{1}{a + b - 1} - \varepsilon,$$

but $\varepsilon > 0$ was arbitrary and so we have the required result. □

## 4.3 Locally $b$-Partite Graphs

In this section we prove Theorem 4.4 in the case $a = 1$, showing that, for $b \geqslant 3$, any locally $b$-partite graph $G$ with minimum degree greater than $(1 - 1/(b + 1/7)) \cdot |G|$ is

$(b+1)$-colourable.

The proof of this (appearing in § 4.3.2) will be an induction upon $b$, with some ideas from the proof of Theorem 3.2 persisting. We first generalise dense and sparse pairs.

**Definition 4.5** ($b$-dense and $b$-sparse)**.** *A pair of* non-adjacent, distinct *vertices $u, v$ in a graph $G$ is $b$-**dense** if $G_{u,v}$ contains a $b$-clique and is $b$-**sparse** if $G_{u,v}$ does not contain a $b$-clique.*

This generalises the notion of dense and sparse given in Definition 3.5 – the definitions given there are identical to those of 2-dense and 2-sparse. Note that any pair of distinct vertices is exactly one of 'adjacent', '$b$-dense', or '$b$-sparse'. Another way to view being $b$-dense is being the missing edge of a $K_{b+2}$. Locally $b$-partite graphs are $K_{b+2}$-free so any pair of distinct vertices with a $b$-clique in their common neighbourhood must be non-adjacent and so dense. In particular, in locally $b$-partite graphs, to establish that a pair is dense does not require checking that they are non-adjacent. As before, we will characterise in what configurations a $b$-sparse pair of vertices can lie. First, we will prove the following lemma which will be helpful for lifting results from the locally bipartite setting.

**Lemma 4.6** (lifting)**.** *Let $b, s$ be positive integers and $\gamma$ any real with $b + \gamma > s$. Let $G$ be a graph with $\delta(G) > (1 - 1/(b+\gamma)) \cdot |G|$. For any $s$-set $X \subset V(G)$, we have*

$$|G_X| \geqslant s\delta(G) - (s-1)|G| > \left(1 - \tfrac{s}{b+\gamma}\right) \cdot |G|, \text{ and}$$
$$\delta(G_X) > \left(1 - \tfrac{1}{b-s+\gamma}\right) \cdot |G_X|.$$

*Proof.* Let $X = \{x_1, \ldots, x_s\}$. Note that for each $v \in V(G)$

$$\mathbb{1}(v \in G_X) \geqslant \mathbb{1}(vx_1 \in E(G)) + \cdots + \mathbb{1}(vx_s \in E(G)) - (s-1),$$

and summing over $v \in V(G)$ gives

$$|G_X| \geqslant s\delta(G) - (s-1)|G| > s\left(1 - \tfrac{1}{b+\gamma}\right)|G| - (s-1)|G| = \left(1 - \tfrac{s}{b+\gamma}\right) \cdot |G|.$$

Note that $\delta(G_X) \geqslant \delta(G) - (|G| - |G_X|) = |G_X| - (|G| - \delta(G))$ so

$$\begin{aligned}\frac{\delta(G_X)}{|G_X|} &\geqslant 1 - \frac{|G| - \delta(G)}{|G|} \cdot \frac{|G|}{|G_X|} > 1 - \left(1 - \frac{\delta(G)}{|G|}\right) \cdot \frac{1}{1 - s/(b+\gamma)} \\ &> 1 - \frac{1}{b+\gamma} \cdot \frac{1}{1 - s/(b+\gamma)} = 1 - \frac{1}{b+\gamma-s}.\end{aligned}$$ □

**Remark 4.7.** If, in Lemma 4.6, $X$ is an $s$-clique and $G$ is locally $b$-partite, then $G_X$ is $(b-s+1)$-colourable and for any $u, v \in G_X$, if the pair $u, v$ is $b$-sparse in $G$, then $u, v$ is $(b-s)$-sparse in $G_X$. (The former can be seen by taking a vertex $x \in X$ and noting that $G_x$ is $b$-colourable and contains the $(s-1)$-clique $X - \{x\}$ joined to $G_X$; the latter by noting that if vertices $u, v \in G_X$ form a $(b-s)$-dense pair in $G_X$, then they form a $b$-dense pair in $G$.)

We given an easy extension to Lemma 3.7 which demonstrates the use of the previous lemma for lifting results.

**Lemma 4.8.** *Let $b \geqslant 2$ be an integer and $G$ be a graph with $\delta(G) > (1-1/b) \cdot |G|$. If $C$ is an induced 4-cycle in $G$ and $G_C$ contains a $(b-2)$-clique, then at least one of the non-edges of $C$ is a $b$-dense pair.*

*Proof.* If $b = 2$, this is just Lemma 3.7. Let $b \geqslant 3$ and $K$ be a $(b-2)$-clique adjacent to all of $C$. Then applying Lemma 4.6 with $X = K$, $s = b-2$, and $\gamma = 0$ gives $\delta(G_K) > (1-1/2) \cdot |G_K| = 1/2 \cdot |G_K|$. Hence, by Lemma 3.7, one of the non-edges of $C$ is a dense pair in $G_K$ so is a $b$-dense pair in $G$. □

The next lemma extends Lemma 3.8 and partly explains why the situation for locally $b$-partite is simpler for $b \geqslant 3$ than for $b = 2$. Lemma 3.8 said that in any locally bipartite $H_0$-free graph, the set of vertices which form a dense pair with a fixed vertex is independent. Here, in place of $H_0$-free, we give a minimum degree condition which guarantees this – for $b \geqslant 3$ this minimum degree condition falls below the chromatic threshold so is, for our purposes, automatic. When $b = 2$, this minimum degree condition is $4/7$, which corresponds to $\overline{C}_7$.

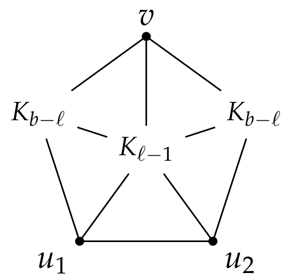
**Lemma 4.9.** *Let $b \geqslant 2$ be an integer and $G$ be a locally $b$-partite graph with $\delta(G) > 2b/(2b+3) \cdot |G|$. For each vertex $v$ of $G$,*

$$D_v := \{u \colon \textit{the pair } u, v \textit{ is } b\textit{-dense}\}$$

*is an independent set of vertices.*

*Proof.* Suppose that in fact there are vertices $v$, $u_1$ and $u_2$ with both pairs $v, u_1$ and $v, u_2$ $b$-dense as well as $u_1$ adjacent to $u_2$. Let $Q_1$ and $Q_2$ be $b$-cliques in $G_{vu_1}$ and $G_{vu_2}$ respectively. Choose $Q_1$ and $Q_2$ so that $\ell = |V(Q_1) \cap V(Q_2)|$ is maximal.

Firstly if $\ell \geqslant 1$, then fix $y \in V(Q_1) \cap V(Q_2)$. Now $G_y$ contains




The pair $v, u_1$ is $(b-1)$-dense in $G_y$ so in any $b$-colouring of $G_y$, $v$ and $u_1$ are the same colour. Similarly in any $b$-colouring of $G_y$, $v$ and $u_2$ are the same colour. In particular, $G_y$ is not $b$-colourable which contradicts the local $b$-colourability of $G$.

Hence $\ell = 0$. Let $X = V(Q_1) \cup V(Q_2) \cup \{v, u_1, u_2\}$ which is a set of $2b+3$ vertices. As $\delta(G) > 2b/(2b+3) \cdot |G|$, some vertex has at least $2b+1$ neighbours in $X$ – call this vertex $x$. As $G_x$ is $K_{b+1}$-free, $x$ has a non-neighbour $x_1 \in V(Q_1) \cup \{u_1\}$ and a non-neighbour $x_2 \in V(Q_2) \cup \{u_2\}$. These must be the only non-neighbours of $x$ in $X$. In particular, $x$ is adjacent to $v$ and so, as $G_x$ is $K_{b+1}$-free, $x_1$ must be in $V(Q_1)$ and $x_2$ must be in $V(Q_2)$. In particular, $Q_1' = Q_1 - \{x_1\} + \{x\}$ is a $b$-clique in $G_{v,u_1}$ and $Q_2' = Q_2 - \{x_2\} + \{x\}$ is a $b$-clique in $G_{v,u_2}$. But $|V(Q_1') \cap V(Q_2')| = 1$ which contradicts the maximality of $\ell$. $\square$

Note that

$$1 - \tfrac{1}{b} \geqslant \tfrac{2b}{2b+3},$$

for all $b \geqslant 3$. We are only interested in locally $b$-partite graphs with $\delta(G) > (1 - 1/b) \cdot |G|$ (the chromatic threshold) and the conclusion of Lemma 4.9 holds for all such graphs.

### 4.3.1 Dismissing Configurations

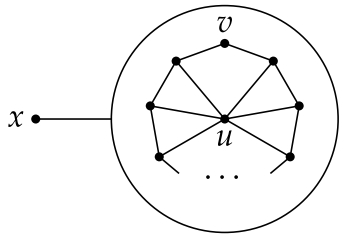
In this subsection, we will rule out various configurations from locally $b$-partite graphs in a similar vein to § 3.5. The motivation for ruling out these particular configurations can be found at the start of our proof in § 4.3.2.

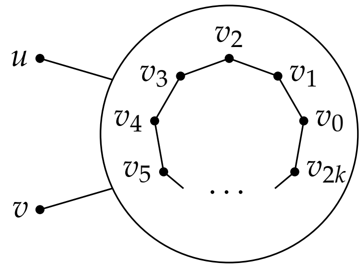
**Proposition 4.10.** *Fix an integer $b \geqslant 3$, let $G$ be a locally $b$-partite graph with $\delta(G) > (1 - 1/(b + 1/7)) \cdot |G|$ and let $u, v$ be a $b$-sparse pair in $G$. Then $G + uv$ does not contain a $K_{b-1} + C_{odd}$ where at least one of $u$, $v$ is in the $K_{b-1}$.*

*Proof.* We will prove this for $b = 3$ and then use the lifting lemma for larger $b$.

For $b = 3$, $G$ is a locally tripartite graph with $\delta(G) > 15/22 \cdot |G| > 2/3 \cdot |G|$ and $u, v$ is a 3-sparse pair in $G$. Suppose the conclusion does not hold. Each neighbourhood of $G$ is 3-colourable so does not contain an odd wheel. In particular, $G$ does not contain

$K_2 + C_{\mathrm{odd}}$, and so $G$ contains one of the following two configurations (corresponding to whether only one of $u, v$ is in the clique or they both are). In the left figure, $x$ is adjacent to all vertices inside the ring, and in the right figure, $u$ and $v$ are adjacent to all vertices inside the ring. In future, we will use rings in this way.




We deal with the left-hand configuration first. Since $G_x$ is 3-colourable, $G_x$ is locally bipartite, $H_0$-free, and $T_0$-free. Applying Lemma 4.6 with $X = \{x\}$ and $\gamma = 1/7$ gives $\delta(G_x) > (1 - 1/(2 + 1/7)) \cdot |G_x| = 8/15 \cdot |G_x|$ so, by Corollary 3.27, $G_x$ does not contain a sparse pair which is the missing spoke of an odd wheel. In particular, the pair $u, v$ is dense in $G_x$ so must be 3-dense in $G$, a contradiction.

Now consider the right-hand configuration. We may assume that $k$ is minimal. As the pair $u, v$ is 3-sparse, $k > 1$. Let $X = \{u, v, v_0, v_1, \ldots, v_{2k}\}$. We consider indices modulo $2k + 1$.

First consider a vertex $x$ adjacent to both $u$ and $v$. We claim that $x$ is adjacent to at most two of the $v_i$ so has at most four neighbours in $X$. For $r \notin \{0, \pm 2\}$, $x$ cannot be adjacent to both $v_i$ and $v_{i+r}$. Indeed if $r = \pm 1$, then $xv_iv_{i+r}$ is a triangle in $G_{u,v}$ while other $r$ give a shorter odd cycle in $G_{u,v}$.

Now consider any other vertex $x$: this is not adjacent to at least one of $u$ or $v$. If this is the only non-neighbour of $x$ in $X$ then $G_x$ contains an odd-wheel, which is not 3-colourable. Thus all vertices have at most $2k + 1$ neighbours in $X$. Hence,

$$\begin{aligned}(2k+3)\delta(G) &\leqslant e(G, X) \leqslant 4|G_{u,v}| + (2k+1)(|G| - |G_{u,v}|) \\ &\leqslant (2k+1)|G| - (2k-3)(2\delta(G) - |G|) = (4k-2)|G| - (4k-6)\delta(G),\end{aligned}$$

which contradicts $\delta(G) > 2/3 \cdot |G|$.

Suppose now that $b \geqslant 4$ and that $G + uv$ does contain a $K_{b-1} + C_{\mathrm{odd}}$ where at least one of $u, v$ is in the $K_{b-1}$. The $(b-1)$-clique contains a $(b-3)$-clique, $L$, with $u, v \notin L$. Thus $L$ is a $(b-3)$-clique in $G$.

Now $G_L$ is a 4-colourable (so locally tripartite) graph in which $u, v$ is 3-sparse (by Remark 4.7). Applying Lemma 4.6 with $X = L$, $\gamma = 1/7$ gives $\delta(G_L) > (1 - 1/(b -$

$(b-3)+1/7)) \cdot |G_L| = 15/22 \cdot |G_L|$. Finally, $G_L + uv$ contains a $K_2 + C_{\mathrm{odd}}$ where at least one of $u$, $v$ is in the 2-clique. This contradicts the result for $b = 3$. □

**Proposition 4.11.** *Fix an integer $b \geqslant 3$, let $G$ be a locally $b$-partite graph with $\delta(G) > (1 - 1/(b + 1/7)) \cdot |G|$ and let $u, v$ be a $b$-sparse pair in $G$. Then $G + uv$ does not contain a $K_{b-2} + H_0$ where at least one of $u$, $v$ is in the $K_{b-2}$.*

*Proof.* We split into two cases depending upon whether only one of $u$, $v$ is in the $(b-2)$-clique or both are. In each case we will prove the result for small $b$ and then use the lifting lemma for larger $b$.

Suppose only one of $u$, $v$ is in the $(b-2)$-clique. We first prove the result for $b = 3$: $G$ is locally tripartite, the pair $u, v$ is 3-sparse, and $\delta(G) > 15/22 \cdot |G| > 2/3 \cdot |G|$. Suppose the conclusion does not hold. Since $G$ does not contain $K_1 + H_0$, $G$ contains the configuration shown in Figure 4.2a where $u$ is adjacent to all of the copy of $H_0$ except for one vertex ($v$) to which is it 3-sparse.


Figure 4.2

Let $X$ be the vertex set of the $H_0$. First consider a vertex $x$ adjacent to $u$: $x$ cannot be adjacent to a triangle or 5-cycle in $G[X]$ otherwise $G + uv$ contains a 2-clique $ux$ joined to an odd cycle which contradicts Proposition 4.10. In particular, any neighbour of $u$ has at most four neighbours in $X$.

Consider any vertex $x$: $\chi(G[X]) = 4$ so $x$ has at most six neighbours in $X$. Hence

$$7\delta(G) \leqslant e(G, X) \leqslant 4d(u) + 6(|G| - d(u)) \leqslant 6|G| - 2\delta(G),$$

which contradicts $\delta(G) > 2/3 \cdot |G|$.

Now let $b \geqslant 4$ and suppose $G + uv$ does contain a $K_{b-2} + H_0$ where exactly one of $u$, $v$ (say $u$) is in the $(b-2)$-clique. Graph $G$ is locally $b$-partite so does not contain $K_{b-2} + H_0$, so $v$ is in the copy of $H_0$. Let $L$ be the $(b-2)$-clique without $u$: $L$ is a $(b-3)$-clique in $G$ and so $G_L$ is 4-colourable and so locally tripartite. Also, by Lemma 4.6, $\delta(G_L) > 15/22 \cdot |G_L|$ and, by Remark 4.7, $u, v$ is a 3-sparse pair in $G_L$. Finally $G_L + uv$ contains $u + H_0$, which contradicts the result we just proved for $b = 3$.

Now consider the second case, where both $u$ and $v$ are in the $(b-2)$-clique: this means that $b \geqslant 4$. We first prove the result for $b = 4$: $G$ is locally 4-partite, the pair $u, v$ is 4-sparse, and $\delta(G) > 22/29 \cdot |G| > 3/4 \cdot |G|$. If the result is false, then $G$ contains the configuration shown in Figure 4.2b.

Let $X = V(H_0) \cup \{u\}$ – a set of eight vertices. First consider a vertex $x$ adjacent to both $u$ and $v$: $x$ cannot be adjacent to a triangle or 5-cycle in $V(H_0)$ as this would contradict Proposition 4.10. Hence $x$ has at most four neighbours in $V(H_0)$ and so at most five in $X$.

Since $G[X]$ is not 4-colourable, all vertices have at most seven neighbours in $X$. Hence

$$\begin{aligned} 8\delta(G) &\leqslant 5|G_{u,v}| + 7(|G| - |G_{u,v}|) \\ &\leqslant 7|G| - 2(2\delta(G) - |G|) = 9|G| - 4\delta(G), \end{aligned}$$

which contradicts $\delta(G) > 3/4 \cdot |G|$.

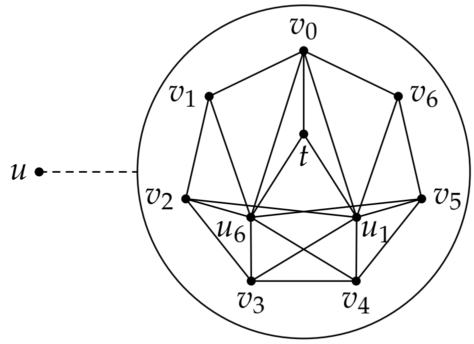
Finally suppose $b \geqslant 5$ and $G + uv$ does contain a $K_{b-2} + H_0$ where both of $u$, $v$ are in the $(b-2)$-clique. Let $L$ be the $(b-2)$-clique without $u$, $v$: $L$ is a $(b-4)$-clique in $G$ so, by Remark 4.7, $G_L$ is 5-colourable and so locally 4-partite. Also, by Lemma 4.6, $\delta(G_L) > 22/29 \cdot |G_L|$ and $u, v$ is a 4-sparse pair in $G_L$. Finally $G_L + uv$ contains $uv + H_0$, which contradicts the result we obtained for $b = 4$. □

**Proposition 4.12.** *Fix an integer $b \geqslant 3$, let $G$ be a locally $b$-partite graph with $\delta(G) > (1 - 1/(b + 1/7)) \cdot |G|$ and let $u, v$ be a $b$-sparse pair in $G$. Then $G + uv$ does not contain a $K_{b-2} + T_0$ where at least one of $u$, $v$ is in the $(b-2)$-clique.*

*Proof.* Again we split into two cases depending upon whether only one of $u$, $v$ is in the $(b-2)$-clique or both are and in each case we will prove the result for small $b$ and then use the lifting lemma for larger $b$.

Suppose only one of $u$, $v$ is in the $(b-2)$-clique. We first prove the result for $b = 3$: $G$ is locally tripartite, the pair $u, v$ is 3-sparse, and $\delta(G) > 15/22 \cdot |G| > 2/3 \cdot |G|$. Suppose the conclusion does not hold. Since $G$ does not contain $K_1 + T_0$, $G$ contains the configuration shown in Figure 4.3a (labels have been added for convenience) where $u$ is adjacent to all of the copy of $T_0$ except for one vertex ($v$) to which it is 3-sparse.

We first show that at least one of the pairs $u_1, v_1$ and $u_6, v_6$ is 3-sparse. Neither of these is an edge as otherwise $G + uv$ contains $u$ joined to a 7-wheel which contradicts Proposition 4.10. If $v \notin \{u_1, u_6, v_3, v_4\}$, then $u_1, u_6$ is 3-dense (triangle $uv_3v_4$) and so $u_1, v_1$ is 3-sparse, by Lemma 4.8. On the other hand, suppose $v \in \{u_1, u_6, v_3, v_4\}$ – by symmetry we may assume $v \neq u_6$. Then $v_1, t$ is 3-dense (triangle $uv_0u_6$) and so $u_1, v_1$ is



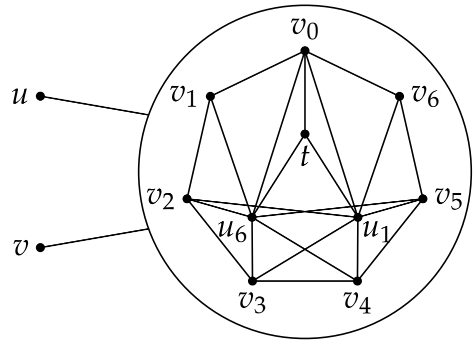




Figure 4.3

3-sparse, by Lemma 4.8. From now on, we will assume that the pair $u_1, v_1$ is 3-sparse in $G$.

Let $G' = G + uv$ – we work in $G'$. From Proposition 4.10, $G'_u$ contains no odd wheel, i.e. is locally bipartite, and from Proposition 4.11, $G'_u$ is $H_0$-free. Now $u_1v_1$ is not an edge (else there is a 7-wheel in $G'_u$) and the pair $v_1, t$ is 2-dense in $G'_u$, so $u_1, v_1$ is 2-sparse in $G'_u$, by Lemma 3.8. In particular, by Lemma 3.20, $u_1v_1$ is not a missing spoke of a 5-wheel in $G'_u$. Hence, by Lemma 3.21, any neighbour of $u_1$ in $G'_u$ is adjacent to at most two of the $v_i$. In particular, any common neighbour of $u$ and $u_1$ in $G'$ is adjacent to at most two of the $v_i$.

Let $X = \{u, u_1, v_0, v_1, \ldots, v_6\}$. What we have just shown is that any common neighbour of $u$ and $u_1$ in $G'$ has at most four neighbours in $X$. Consider a vertex $x$ which is not adjacent to both $u$ and $u_1$: $x$ cannot be adjacent to all of $X \setminus \{u_1\}$ otherwise $G'_{u,x}$ contains a 7-cycle which contradicts Proposition 4.10. Also $x$ cannot be adjacent to all of $X \setminus \{u\}$ as otherwise $G'_x = G_x$ contains a 7-wheel missing the spoke $u_1v_1$ which is 3-sparse in $G$. This again contradicts Proposition 4.10. Hence, any vertex has at most seven neighbours in $X$. Thus,

$$\begin{aligned} 9\delta(G) &\leqslant 9\delta(G') \leqslant 4|G'_{u,u_1}| + 7(|G'| - |G'_{u,u_1}|) = 7|G| - 3|G'_{u,u_1}| \\ &\leqslant 7|G| - 3|G_{u,u_1}| \leqslant 7|G| - 3(2\delta(G) - |G|) = 10|G| - 6\delta(G), \end{aligned}$$

which contradicts $\delta(G) > 2/3 \cdot |G|$.

Now let $b \geqslant 4$ and suppose $G + uv$ does contain a $K_{b-2} + T_0$ where exactly one of $u$, $v$ (say $u$) is in the $(b-2)$-clique. Graph $G$ is locally $b$-partite so does not contain $K_{b-2} + T_0$ so $v$ is in the copy of $T_0$. Let $L$ be the $(b-2)$-clique without $u$: $L$ is a $(b-3)$-clique in $G$ and so $G_L$ is 4-colourable and so locally tripartite. Also, by Lemma 4.6, $\delta(G_L) > 15/22 \cdot |G_L|$ and $u, v$ is a 3-sparse pair in $G_L$. Finally $G_L + uv$ contains $u + T_0$, which contradicts the result we just proved for $b = 3$.

Now consider the second case, where both $u$ and $v$ are in the $(b-2)$-clique: this means that $b \geqslant 4$. We first prove the result for $b = 4$: $G$ is locally 4-partite, the pair $u, v$ is 4-sparse, and $\delta(G) > 22/29 \cdot |G| > 3/4 \cdot |G|$. If the result is false, then $G$ contains the configuration shown in Figure 4.3b (labels have been added for convenience).

By Propositions 4.10 and 4.11, $G_{u,v}$ is locally bipartite and $H_0$-free. By Lemma 4.6, $\delta(G_{u,v}) > 1/2 \cdot |G_{u,v}|$. Since $G_{u,v}$ does not contain an odd wheel, $u_1v_1$ is not an edge. Also $v_1, t$ is a 2-dense pair, $tu_1$ is an edge, and $G_{u,v}$ is $H_0$-free, so by Lemma 3.8 $u_1, v_1$ is a 2-sparse pair in $G_{u,v}$. Similarly $u_6, v_6$ is 2-sparse in $G_{u,v}$.

Now let $X = \{u, u_1, u_6, v_0, \ldots, v_6\}$ (note that this does not contain $t$ or $v$). Within $G_{u,v}$, $u_1$ together with the $v_i$ form a 7-wheel missing the spoke $u_1v_1$ which is a sparse pair. Since $G_{u,v}$ is locally bipartite with $\delta(G_{u,v}) > 1/2 \cdot |G_{u,v}|$, Lemma 3.20 implies that $G_{u,v}$ does not contain any 5-wheels missing a sparse spoke. Hence, by Lemma 3.21, any neighbour of $u_1$ in $G_{u,v}$ is adjacent to at most two of the $v_i$. Thus, any neighbour of $u$, $v$, $u_1$ has at most five neighbours in $X$ (two amongst $v_i$ together with possibly $u_1, u_6, u$). Similarly any neighbour of $u$, $v$, $u_6$ has at most five neighbours in $X$. Next consider a vertex $x$ adjacent to both $u$, $v$ but to neither $u_1$ nor $u_6$. As $G_{u,v}$ is locally bipartite, $x$ is adjacent to at most six of the $v_i$ so $x$ has at most seven neighbours in $X$. Finally $\chi(G[X]) = 5$ so all vertices have at most nine neighbours in $X$. Hence

$$\begin{aligned}
10\delta(G) &\leqslant 5|\Gamma(u,v,u_1) \cup \Gamma(u,v,u_6)| + 7(|G_{u,v}| - |\Gamma(u,v,u_1) \cup \Gamma(u,v,u_6)|) \\
&\qquad + 9(|G| - |G_{u,v}|) \\
&= 9|G| - 2|G_{u,v}| - 2|\Gamma(u,v,u_1) \cup \Gamma(u,v,u_6)| \leqslant 9|G| - 2|G_{u,v}| - 2|G_{u,v,u_1}| \\
&\leqslant 9|G| - 2(2\delta(G) - |G|) - 2(3\delta(G) - 2|G|) = 15|G| - 10\delta(G),
\end{aligned}$$

which contradicts $\delta(G) > 3/4 \cdot |G|$.

Now let $b \geqslant 5$ and suppose $G + uv$ does contain a $K_{b-2} + T_0$ where both of $u, v$ are in the $(b-2)$-clique. Let $L$ be the $(b-2)$-clique without $u, v$: $L$ is a $(b-4)$-clique in $G$ and so $G_L$ is 5-colourable and so locally 4-partite. Also, by Lemma 4.6, $\delta(G_L) > 22/29 \cdot |G_L|$ and, by Remark 4.7, the pair $u, v$ is 4-sparse in $G_L$. Finally $G_L + uv$ contains $uv + T_0$, which contradicts the result just proved for $b = 4$. □

## 4.3.2 Finishing the Proof

Here we will prove Theorem 4.4 for locally $b$-partite graphs.

*Proof.* Take an edge-maximal locally $b$-partite graph $G$ with $\delta(G) > (1 - 1/(b + 1/7)) \cdot |G|$. We need to show that $G$ is $(b+1)$-colourable. We may assume by induction that the theorem holds for all $b'$ with $3 \leqslant b' < b$ (if there are any).

We first show that for any $b$-sparse pair $u, v$ of $G$, $G' = G + uv$ contains a $(b-2)$-clique $K$ with $G'_K$ not 3-colourable. Indeed, for $b = 3$, $G'$ is not locally tripartite (by edge-maximality) so there is a vertex in $G'$ whose neighbourhood is not 3-colourable. Take $K$ to be this vertex. For $b > 3$, $G'$ is not locally $b$-partite so contains a vertex $w_1$ with $G'_{w_1}$ not $b$-colourable. Applying Lemma 4.6 with $X = \{w_1\}$ and $\gamma = 1/7$ gives

$$\delta(G'_{w_1}) > \left(1 - \frac{1}{b - 1 + 1/7}\right) \cdot |G'_{w_1}|.$$

By the induction hypothesis, if $G'_{w_1}$ was locally $(b-1)$-partite, then it would be $b$-colourable. In particular, $G'_{w_1}$ is not locally $(b-1)$-partite and so there is a vertex $w_2$ in $G'_{w_1}$ with $G'_{w_1,w_2}$ not $(b-1)$-colourable. Repeating this argument gives a $(b-2)$-clique $K$ with $G'_K$ not 3-colourable.

Now, applying Lemma 4.6 with $X = K$ and $\gamma = 1/7$ gives

$$\delta(G'_K) > \left(1 - \frac{1}{b - (b-2) + 1/7}\right) \cdot |G'_K| = 8/15 \cdot |G'_K|.$$

By Theorem 3.2 and Theorem 3.10, $G'_K$ contains either an odd wheel, a copy of $H_2$, or a copy of $T_0$. Hence $G'$ contains either $K_{b-2} + W_{\text{odd}} = K_{b-1} + C_{\text{odd}}$, $K_{b-2} + H_2$, or $K_{b-2} + T_0$. Note that $G$ cannot contain any of these so $uv$ is a missing edge from one of these configurations. Propositions 4.10 to 4.12 mean that both $u$ and $v$ lie in the $C_{\text{odd}}$, the $H_2$, or the $T_0$.

In particular, $u, v \notin K$ so $K$ is a $(b-2)$-clique in $G$ and $V(G_K) = V(G'_K)$. We have the following facts.

- By Remark 4.7, $G_K$ is 3-colourable and so locally bipartite.
- By Remark 4.7, $u, v$ is a 2-sparse pair in $G_K$.
- Applying Lemma 4.6 with $X = K$ and $\gamma = 1/7$ gives $\delta(G_K) > 8/15 \cdot |G_K|$.
- The graph $G_K$ contains no odd wheel, $H_0$, or $T_0$ ($G_K$ is 3-colourable) but the addition of $uv$ introduces an odd wheel, a copy of $H_2$, or a copy of $T_0$.

Using the argument at the start of § 3.6, we deduce that, within $G_K$, there must be one of the configurations appearing in Figure 3.5 on page 53 (with labels $u$ and $v$ possibly swapped). Note in that proof we only used edge-maximality to show that $uv$ was the missing edge of an odd wheel, a copy of $H_2$, or a copy of $T_0$ (and so here we do not need $G_K$ to be an edge-maximal locally bipartite graph).

We now mimic the remainder of the proof of Theorem 3.2. Let $I$ be a largest independent set in $G$: $|I| = \alpha(G)$. Recall for a vertex $u$ that

$$D_u = \{v \colon \text{the pair } u, v \text{ is } b\text{-dense}\}.$$

**Proposition 4.13.** *For all distinct $u, v \in I$, the pair $u, v$ is $b$-dense and furthermore for every $u \in I$, $D_u \cup \{u\} = I$.*

*Proof.* Fix distinct $u, v \in I$. We will first show that $G_{u,v}$ is not $(b-1)$-colourable. Note that $\Gamma(u), \Gamma(v) \subset V(G) \setminus I$ so $|\Gamma(u) \cup \Gamma(v)| \leqslant |G| - |I|$. Also $I \subset V(G) \setminus \Gamma(u)$, so $|I| \leqslant |G| - d(u) \leqslant |G| - \delta(G)$. Hence,

$$\begin{aligned}|\Gamma(u,v)| &= d(u) + d(v) - |\Gamma(u) \cup \Gamma(v)| \geqslant 2\delta(G) + |I| - |G| \\ &= b\delta(G) - (b-1)|G| + (b-2)(|G| - \delta(G)) + |I| \\ &\geqslant b\delta(G) - (b-1)|G| + (b-1)|I| > (b-1)|I|,\end{aligned}$$

where we used $\delta(G) > (1 - 1/b) \cdot |G|$ in the final inequality. But $I$ was a largest independent set in $G$ so $G_{u,v}$ cannot be covered by $b-1$ independent sets and hence is not $(b-1)$-colourable.

Now we will show that $u, v$ is $b$-dense. Suppose not and so they form a $b$-sparse pair. If $b = 3$, then $G_{u,v}$ is not bipartite and so contains an odd cycle. This contradicts Proposition 4.10. For $b \geqslant 4$ we will find a $(b-4)$-clique $K$ in $G_{u,v}$ with $G_{u,v,K}$ not 3-colourable. For $b = 4$, we take $K = \varnothing$ and this suffices. For larger $b$, we note that, by Lemma 4.6, $\delta(G_{u,v}) > (1 - 1/(b - 2 + 1/7)) \cdot |G_{u,v}|$. By the induction hypothesis, if $G_{u,v}$ were locally $(b-2)$-partite, then $G_{u,v}$ would be $(b-1)$-colourable, which it is not. Thus, there is $w_1 \in G_{u,v}$ with $G_{u,v,w_1}$ not $(b-2)$-colourable. Repeating this argument we obtain a $(b-4)$-clique $K$ with $G_{u,v,K}$ not 3-colourable. Applying Lemma 4.6 with $X = \{u, v, K\}$ and $\gamma = 1/7$ gives

$$\delta(G_{u,v,K}) > (1 - 1/(b - (b-2) + 1/7)) \cdot |G_{u,v,K}| = 8/15 \cdot |G_{u,v,K}|.$$

Then Theorem 3.9 gives that $G_{u,v,K}$ either contains an odd wheel, a copy of $H_0$, or a copy of $T_0$. These contradict Propositions 4.10 to 4.12 (applied to $G$). We have shown that $u, v$ must be $b$-dense. Thus $I \subset D_u \cup \{u\}$.

On the other hand, by the definition of density and Lemma 4.9, $D_u \cup \{u\}$ is an independent set. However, it contains the maximal independent set $I$ so must equal it. □

**Definition 4.14** ($b$-quasidense)**.** *A pair of vertices $u, v$ is $b$-**quasidense** if there is a sequence of vertices $u = d_1, d_2, \ldots, d_k, d_{k+1} = v$ such that all pairs $d_i, d_{i+1}$ are $b$-dense $(i = 1, 2, \ldots, k)$.*

Proposition 4.13 immediately implies that if $u, v$ is $b$-quasidense and $u \in I$, then $v \in I$ as well. Now we can finish the proof. It suffices to show that every vertex is either in $I$ or is adjacent to all of $I$. Indeed, we may then fix $u \in I$ and note that $G[V(G) \setminus I] = G_u$

so $G[V(G) \setminus I]$ is $b$-colourable. Using a further colour for the independent set $I$ gives a $(b+1)$-colouring of $G$.

Suppose instead there is $u \in I$ and $v \notin I$ with $u$ not adjacent to $v$. In particular, the pair $u, v$ cannot be $b$-quasidense and so is $b$-sparse. Thus from our remarks just preceding Proposition 4.13, there is a $(b-2)$-clique $K$ in $G$ such that $G_K$ contains one of the configurations appearing in Figure 3.5 on page 53 (with labels $u$ and $v$ possibly swapped) and the pair $u, v$ is 2-sparse in $G_K$.

Focus on $G_K$ – this graph is 3-colourable so locally bipartite, $H_0$-free, and $T_0$-free and the pair $u, v$ is 2-sparse in $G_K$. Also, by Lemma 4.6, $\delta(G_K) > (1 - 1/(b - (b-2) + 1/7)) \cdot |G_K| = 8/15 \cdot |G_K|$. In the proof of Proposition 3.30, we used these facts alone to show that $u, v$ is quasidense in every configuration appearing in Figure 3.5. Hence the pair $u, v$ is quasidense in $G_K$ so is $b$-quasidense in $G$. This is our required contradiction. □

## 4.4 $a$-Locally $b$-Partite Graphs

In this section we relate the chromatic profile of $a$-locally $b$-partite graphs to the chromatic profile of locally $b$-partite graphs, making precise our comment in the introduction of this chapter that to understand $a$-locally $b$-partite graphs it seems to be enough to understand locally $b$-partite graphs. This is elucidated at the end of § 4.4.1 and just before Theorem 4.20. Along the way we will prove Theorems 4.3 and 4.4.

### 4.4.1 The First Threshold – Proving Theorems 4.3 and 4.4

As noted in the introduction, the first interesting threshold is $\delta_\chi(\mathcal{F}_{a,b}, a+b)$ – what values of $c$ guarantee that any $a$-locally $b$-partite graph with $\delta(G) \geqslant c|G|$ is $(a+b)$-colourable? We already know $\delta_\chi(\mathcal{F}_{1,2}, 3) = 4/7$ and $\delta_\chi(\mathcal{F}_{1,b}, b+1) \leqslant 1 - 1/(b + 1/7)$ and will extend these to all values of $a$. To simplify the statements of our results and make comparisons between different values of $a$ and $b$, it is helpful to write

$$\delta_\chi(\mathcal{F}_{a,b}, a+b) = 1 - \frac{1}{a+b-1+\gamma_{a,b}},$$

and to focus our attention on the $\gamma_{a,b}$. As $\delta_\chi(\mathcal{F}_{a,b}, a+b) \geqslant \delta_\chi(\mathcal{F}_{a,b})$ we have, from Theorem 4.2,

$$\gamma_{a,b} \geqslant 0.$$

We collect some other basic properties of the $\gamma_{a,b}$.

**Lemma 4.15.** *For all positive integers $a$ and $b$ the following hold.*

- $\delta_\chi(\mathcal{F}_{a,b+1}, a+b) \leqslant \delta_\chi(\mathcal{F}_{a+1,b}, a+b)$ *and so*

$$\gamma_{a,b+1} \leqslant \gamma_{a+1,b}. \tag{4.1}$$

- $1/(2-\delta_\chi(\mathcal{F}_{a,b}, a+b)) \leqslant \delta_\chi(\mathcal{F}_{a+1,b}, a+b+1)$ *and so*

$$\gamma_{a,b} \leqslant \gamma_{a+1,b}. \tag{4.2}$$

*Also* $\gamma_{1,2} = 1/3$ *and* $\gamma_{1,b} \leqslant 1/7$ *for all* $b \geqslant 3$.

*Proof.* We showed in Chapter 3 that $\delta_\chi(\mathcal{F}_{1,2}, 3) = 4/7$ giving $\gamma_{1,2} = 1/3$ while, for $b \geqslant 3$, § 4.3 showed $\delta_\chi(\mathcal{F}_{1,b}, b+1) \leqslant 1 - 1/(b+1/7)$ so $\gamma_{1,b} \leqslant 1/7$.

Now $\mathcal{F}_{a,b+1} \subset \mathcal{F}_{a+1,b}$ from which $\delta_\chi(\mathcal{F}_{a,b+1}, a+b) \leqslant \delta_\chi(\mathcal{F}_{a+1,b}, a+b)$ immediately follows. This gives inequality (4.1).

Finally let $d < \delta_\chi(\mathcal{F}_{a,b}, a+b)$: there is an $a$-locally $b$-partite graph $G$ with $\delta(G) \geqslant d|G|$ and $\chi(G) > a+b$. Let $G'$ be $G$ joined to an independent set of size $|G| - \delta(G)$, that is,

$$G' = K_1(|G| - \delta(G)) + G.$$

Since $G$ is $a$-locally $b$-partite, it is also $(a+1)$-locally $(b-1)$-partite. From both of these it follows that $G'$ is $(a+1)$-locally $b$-partite. Also, $\chi(G') = \chi(G) + 1 > a+b+1$, and

$$\frac{\delta(G')}{|G'|} = \frac{|G|}{2|G| - \delta(G)} = \frac{1}{2 - \delta(G)|G|^{-1}} \geqslant \frac{1}{2-d},$$

so $\delta_\chi(\mathcal{F}_{a+1,b}, a+b+1) \geqslant 1/(2-d)$. This holds for all $d < \delta_\chi(\mathcal{F}_{a,b}, a+b)$, so

$$\delta_\chi(\mathcal{F}_{a+1,b}, a+b+1) \geqslant \frac{1}{2 - \delta_\chi(\mathcal{F}_{a,b}, a+b)}.$$

Thus

$$1 - \frac{1}{a+b+\gamma_{a+1,b}} \geqslant \frac{1}{1 + \frac{1}{a+b-1+\gamma_{a,b}}} = \frac{a+b-1+\gamma_{a,b}}{a+b+\gamma_{a,b}} = 1 - \frac{1}{a+b+\gamma_{a,b}},$$

and so $\gamma_{a+1,b} \geqslant \gamma_{a,b}$, as required. □

Inequality (4.2) gives a lower bound for $\gamma_{a+1,b}$ in terms of $\gamma_{a,b}$. The next lemma, which lies at the heart of our analysis, gives an upper bound.

**Lemma 4.16.** *For all positive integers $a$ and $b$,*

$$\gamma_{a,b} \leqslant \gamma_{a+1,b} \leqslant \max\{\gamma_{a,b}, \gamma_{1,a+b}\}. \tag{4.3}$$

*Proof.* The left-hand inequality is just inequality (4.2). Let $\gamma = \max\{\gamma_{a,b}, \gamma_{1,a+b}\}$. Let $G$ be an $(a+1)$-locally $b$-partite graph with

$$\delta(G) > \left(1 - \frac{1}{a+b+\gamma}\right) \cdot |G|.$$

It suffices to show that $\chi(G) \leqslant a+b+1$ as then

$$1 - \frac{1}{a+b+\gamma} \geqslant \delta_\chi(\mathcal{F}_{a+1,b}, a+b+1) = 1 - \frac{1}{a+b+\gamma_{a+1,b}}.$$

Fix any $u \in V(G)$ and consider $G_u$: $G_u$ is an $a$-locally $b$-partite graph with

$$\delta(G_u) > \left(1 - \frac{1}{a+b-1+\gamma}\right) \cdot |G_u|,$$

by the lifting lemma, Lemma 4.6. But $\gamma \geqslant \gamma_{a,b}$ so

$$\delta(G_u) > \delta_\chi(\mathcal{F}_{a,b}, a+b) \cdot |G_u|,$$

and hence $G_u$ is $(a+b)$-colourable. Thus $G$ is a locally $(a+b)$-partite graph. Also $\gamma \geqslant \gamma_{1,a+b}$, so

$$\delta(G) > \delta_\chi(\mathcal{F}_{1,a+b}, a+b+1) \cdot |G|.$$

Thus $G$ is $(a+b+1)$-colourable. □

From this one can immediately deduce Theorems 4.3 and 4.4.

**Corollary 4.17** (Theorems 4.3 and 4.4)**.** *For all positive integers $a$ and for all $b \geqslant 3$,*

$$\gamma_{a,2} = 1/3, \quad \gamma_{a,b} \leqslant 1/7,$$

*and so*

$$\delta_\chi(\mathcal{F}_{a,2}, a+2) = 1 - \frac{1}{a+1+1/3}, \qquad \delta_\chi(\mathcal{F}_{a,b}, a+b) \leqslant 1 - \frac{1}{a+b-1+1/7}.$$

*Proof.* From Lemma 4.15, $\gamma_{1,2} = 1/3$ and $\gamma_{1,b} \leqslant 1/7$ for any $b \geqslant 3$. By Lemma 4.16, for any $a$ and any $b \geqslant 2$,

$$\gamma_{a,b} \leqslant \gamma_{a+1,b} \leqslant \max\{\gamma_{a,b}, \gamma_{1,a+b}\} \leqslant \max\{\gamma_{a,b}, 1/7\}.$$

An easy induction gives $\gamma_{a,2} = 1/3$ for all $a$ and $\gamma_{a,b} \leqslant 1/7$ for any $b \geqslant 3$. □

Note that inequalities (4.1) and (4.2) give

$$\gamma_{a,b} \geqslant \gamma_{1,b'},$$

for all $b \leqslant b' \leqslant a+b-1$. In line with our inductive arguments in § 4.3, we believe that in fact $\gamma_{1,b} \geqslant \gamma_{1,b+1}$ for all $b$. If this were true, then $\gamma_{a,b} \geqslant \gamma_{1,a+b}$, and so Lemma 4.16 would give $\gamma_{a,b} = \gamma_{a+1,b}$ for all $a$, $b$. Of course, this implies $\gamma_{a,b} = \gamma_{1,b}$ and so $\delta_\chi(\mathcal{F}_{a,b}, a+b)$ would be determined by $\delta_\chi(\mathcal{F}_{1,b}, b+1)$ – a particular manifestation of our aforementioned belief that to understand $a$-locally $b$-partite graphs, we should focus on locally $b$-partite graphs. It also highlights the following question.

**Question 4.18.** *Is the sequence $\gamma_{1,b}$ decreasing in $b$?*

### 4.4.2 $a$-Locally Bipartite Graphs

One could replicate the elementary approach of the previous section to try to evaluate $\delta_\chi(\mathcal{F}_{a,b}, k)$ for $k > a+b$. Indeed one might define $\gamma_{a,b,m}$ by

$$\delta_\chi(\mathcal{F}_{a,b}, a+b+m) = 1 - \frac{1}{a+b-1+\gamma_{a,b,m}},$$

so that $\gamma_{a,b,0} = \gamma_{a,b}$. Many of the properties of the $\gamma_{a,b}$ pass over: the $\gamma_{a,b,m}$ are non-negative (and, in fact, $\lim_{m\to\infty} \gamma_{a,b,m} = 0$) and both inequalities (4.1) and (4.2) extend easily ($\gamma_{a,b+1,m} \leqslant \gamma_{a+1,b,m}$ and $\gamma_{a,b,m} \leqslant \gamma_{a+1,b,m}$). However, there seems to be no argument to produce inequality (4.3) or anything similar. A more involved approach would be required.

The next threshold to consider is $\delta_\chi(\mathcal{F}_{a,b}, a+b+1)$. For locally bipartite graphs, we showed $\delta_\chi(\mathcal{F}_{1,2}, 4) \leqslant 6/11$ and had many structural results (some of which we will extend). For $b \geqslant 3$, we know very little about $\delta_\chi(\mathcal{F}_{1,b}, b+2)$ beyond it being at least $\delta_\chi(\mathcal{F}_{1,b}) = 1 - 1/b$ and at most $\delta_\chi(\mathcal{F}_{1,b}, b+1) \leqslant 1 - 1/(b+1/7)$. The question for $b = 3$ is of particular interest. Tantalisingly, the complement of the 9-cycle is locally tripartite, 5-chromatic and has minimum degree $6 = 2/3 \cdot 9$.

**Question 4.19.** *Is there a locally tripartite graph $G$ with minimum degree greater than $2/3 \cdot |G|$ that is not 4-colourable?*

We now focus on $a$-locally bipartite graphs. The following theorem, which will be essential for extending the Andrásfai-Erdős-Sós theorem in Chapter 5, should be compared to Theorem 3.2 – again we see that the key to understanding $a$-locally bipartite graphs is to understand locally bipartite ones. The proof is an induction combining our results for locally bipartite (Theorem 3.2) and locally $b$-partite (Theorem 4.4) graphs.

**Theorem 4.20** ($a$-locally bipartite graphs)**.** *Let $G$ be an $a$-locally bipartite graph.*

- *If $\delta(G) > (1 - 1/(a + 4/3)) \cdot |G|$, then $G$ is $(a + 2)$-colourable. Blow-ups of $K_{a-1} + \overline{C}_7$ show that this is tight.*
- *If $\delta(G) > (1 - 1/(a + 5/4)) \cdot |G|$, then $G$ is either $(a + 2)$-colourable or contains $K_{a-1} + \overline{C}_7$.*
- *If $\delta(G) > (1 - 1/(a + 6/5)) \cdot |G|$, then $G$ is either $(a + 2)$-colourable or contains $K_{a-1} + \overline{C}_7$ or $K_{a-1} + H_2^+$.*
- *If $\delta(G) > (1 - 1/(a + 7/6)) \cdot |G|$, then $G$ is either $(a + 2)$-colourable or contains $K_{a-1} + H_2$.*
- *If $\delta(G) > (1 - 1/(a + 8/7)) \cdot |G|$, then $G$ is either $(a + 2)$-colourable or contains $K_{a-1} + H_2$ or $K_{a-1} + T_0$.*

*Proof.* The suitable blow-ups mentioned in the first bullet point are balanced blow-ups of $K_{a-1}(3) + \overline{C}_7$. Proving everything else is a simple induction on $a$ (with Theorem 3.2 covering the base case). Indeed we will just demonstrate it for the final bullet point. Let $G$ be an $a$-locally bipartite graph with $\delta(G) > (1 - 1/(a + 8/7)) \cdot |G|$. Fix any vertex $u$ of $G$ and consider $G_u$: this is $(a - 1)$-locally bipartite and by Lemma 4.6, $\delta(G_u) > (1 - 1/(a - 1 + 8/7)) \cdot |G_u|$. By induction, either $G_u$ contains one of $K_{a-2} + H_2$, $K_{a-2} + T_0$ or is $(a + 1)$-colourable. If there is some vertex $u$ with $G_u$ not $(a + 1)$-colourable, then $G$ contains one of $K_{a-1} + H_2$, $K_{a-1} + T_0$. Otherwise, $G$ is locally $(a + 1)$-partite. But, by Theorem 4.4,

$$\delta_\chi(\mathcal{F}_{1,a+1}, a + 2) \leqslant 1 - \frac{1}{a + 8/7},$$

so $G$ is $(a + 2)$-colourable, as required. □

# CHAPTER 5

# MINIMUM DEGREE STABILITY

Many of the results of this chapter have been submitted in a forthcoming paper [Ill21c].

## 5.1 INTRODUCTION

Consider again the Andrásfai-Erdős-Sós theorem, Theorem 1.2. In the previous two chapters, we have viewed this as the first step in the chromatic profile of $K_{r+1}$-free graphs. However, it is far more than that and, in this chapter, we view it through the lens of stability. The results of Erdős and Simonovits, Theorem 1.1, act as an edge stability result for $H$-free graphs (where $H$ is some fixed graph of chromatic number $r+1$): any $H$-free graph with $(1-1/r+o(1))\binom{n}{2}$ edges is close to (within $o(n^2)$ edges of) being $r$-partite. Furthermore, the discussion opening § 1.1.2 shows that $1-1/r$ cannot be replaced by any smaller constant. Andrásfai, Erdős, and Sós's theorem demonstrates that quite a different phenomenon occurs for the minimum degree stability of $K_{r+1}$-free graphs. The minimum degree can be as low as $(1-1/(r-1/3))n$ in a $K_{r+1}$-free and it still must remain $r$-partite (recall that the $K_{r+1}$-free graph with greatest minimum degree, $T_r(n)$, has minimum degree $(1-1/r)n$). The tightness of $1-1/(r-1/3)$ in Theorem 1.2 is shown by $K_{r-2}(3)+C_5$.

Moreover, $1-1/(r-1/3)$ is tight for the minimum degree stability question too: $n$-vertex balanced blow-ups of $K_{r-2}(3)+C_5$ are $K_{r+1}$-free, $(r+1)$-chromatic, have minimum degree $\lfloor(1-1/(r-1/3))n\rfloor$, and require the deletion of $\Omega(n^2)$ edges to be made $r$-partite as shown by Lemma 1.7. This suggests the most basic problem for the minimum degree stability of $H$-free graphs: given an $(r+1)$-chromatic graph $H$, determine

$$\begin{aligned}\delta_H = \inf\{c\colon &\text{ if } |G|=n,\ \delta(G)\geqslant cn, \text{ and } G \text{ is } H\text{-free},\\ &\text{then } G \text{ can be made } r\text{-partite by deleting } o(n^2) \text{ edges}\}.\end{aligned}$$

This is the analogue of the edge stability question answered by Theorem 1.1. The reason for allowing the omission of $o(n^2)$ edges is the same for both questions: there are many $H$ for which the $H$-free graph with most edges or greatest minimum degree is not $r$-partite (but it must always be close to – it is close to $T_r(n)$). This is explored further in § 5.1.1.

It follows from our previous discussion that

$$\delta_{K_{r+1}} = 1 - \frac{1}{r - 1/3}.$$

As pointed out by Alon and Sudakov [AS06], a standard application of Szemerédi's regularity lemma [Sze78] shows that

$$\delta_{K_{r+1}(t)} = 1 - \frac{1}{r - 1/3}.$$

Alon and Sudakov showed that one can do better than deleting $o(n^2)$ edges.

**Theorem 5.1** (Alon-Sudakov)**.** *Let $r \geqslant 2$ and $t \geqslant 1$ be integers, let $\varepsilon > 0$, and set $\rho = 1/(4r^{2/3}t)$. The following holds for all sufficiently large $n$. If $G$ is a $K_{r+1}(t)$-free graph on $n$ vertices and with minimum degree at least $(1 - 1/(r - 1/3) + \varepsilon)n$, then one can delete $\mathcal{O}_{r,t,\varepsilon}(n^{2-\rho})$ edges to make $G$ $r$-colourable.*

Subsequently, Allen [All10] found a more direct proof (with no use of the regularity lemma) that yields optimal (to within a constant factor) estimates in terms of degenerate Turán numbers for the number of edges that need deleting. Given the central role played by $K_{r+1}(t)$ in edge stability, it might be tempting to believe that this should determine $\delta_H$ for general non-bipartite $H$. Of course, if $H$ is a graph with chromatic number $r + 1$, then $H$ is a subgraph of $K_{r+1}(t)$ for some $t$ and so any $H$-free graph is $K_{r+1}(t)$-free. Thus

$$\delta_H \leqslant \delta_{K_{r+1}(t)} = 1 - \frac{1}{r - 1/3}, \tag{5.1}$$

and the improved bounds of Theorem 5.1 apply (Allen's bounds are tight up to a constant). Nonetheless, the inequality may be strict. In the case of edge stability, the Turán graph $T_r(n)$ has $(1 - 1/r)\binom{n}{2}$ edges and does not contain $H$, as $H$ has chromatic number $r + 1$. Here, however, it is blow-ups of $K_{r-2}(3) + C_5$ which show that $\delta_{K_{r+1}(t)} \geqslant 1 - 1/(r - 1/3)$. These are $K_{r+1}$-free, but need not be $H$-free – for a simple example, consider $r = 2$ and $H$ an odd cycle. Minimum degree stability is more nuanced: $\delta_H$ is determined not just by the chromatic number of $H$ but also by its finer structural properties. For 3-chromatic graphs, the situation is fairly straightforward.

**Theorem 5.2** ($\delta_H$ for 3-chromatic $H$)**.** *Let $H$ be a 3-chromatic graph. There is a smallest positive integer $g$ such that $H$ is not homomorphic to $C_{2g+1}$. Then*

$$\delta_H = \frac{2}{2g+1}.$$

Thus, for 3-chromatic $H$, $\delta_H$ is determined by the first odd cycle to which $H$ is not homomorphic. Next we turn to graphs that are not 3-colourable. We will determine $\delta_H$ for very many $H$ (indeed, all that one is commonly interested in) and bound it for the remainder.

**Theorem 5.3** ($\delta_H$ for $H$ not 3-colourable)**.** *There exists a sequence of eleven graphs $(F_g)_{1\leqslant g\leqslant 11}$ and constants $(c_g)_{1\leqslant g\leqslant 11}$ (described explicitly in § 5.1.2) such that the following holds. Let $r \geqslant 3$ be an integer. If $H$ is an $(r+1)$-chromatic graph and $g$ is minimal such that $H$ is not homomorphic to $K_{r-3} + F_g$, then*

$$\delta_H = 1 - \frac{1}{r - 1 + c_g}.$$

*If $H$ is homomorphic to all eleven $K_{r-3} + F_g$, then there is a least $g$ for which $H$ is not homomorphic to $K_{r-2} + C_{2g+1}$: $\delta_H$ satisfies the bounds*

$$1 - \frac{1}{r-1} < 1 - \frac{1}{r - 1 + 2/(2g-1)} \leqslant \delta_H \leqslant 1 - \frac{1}{r - 1 + 1/7}.$$

As for 3-colourable $H$, there is a sequence of graphs such that $\delta_H$ is determined by the first one to which $H$ is not homomorphic. We will define the $F_g$ and $c_g$ explicitly in § 5.1.2 – the reader will recognise them as the graphs that appeared in our results concerning locally bipartite graphs.

The only $(r+1)$-chromatic graphs whose $\delta_H$ value is not determined by Theorem 5.3 are those which are homomorphic to all eleven $K_{r-3} + F_g$. Such a graph would be have a very precise structure indeed.

### 5.1.1 Wider Context – the Theory of Chromatic Profiles

We give a second view of $\delta_H$, placing it within a whole spectrum of structural constants relating to $H$-free graphs, and address some subtleties of the Erdős-Simonovits problem for $H$-free graphs including why it might not quite be the right question. We remind the reader that the chromatic profile of a graph $H$ is the sequence of values (for $k = 1, 2, \dots$) of

$$\delta_\chi(H,k) = \inf\{c\colon \text{if } \delta(G) \geqslant c|G| \text{ and } G \text{ is } H\text{-free, then } G \text{ is } k\text{-colourable}\}.$$

We now address an issue that we have thus far skirted. As mentioned previously, very little is known about the chromatic profile for general $H$ with Erdős and Simonovits [ES73] describing it as 'too complicated'. In part this is because the exact structure and chromatic number of the $n$-vertex $H$-free graphs with highest minimum degree is unknown. Moreover, degenerate examples abound. Consider $H = K_3(2)$ and let $G$ be the complete bipartite Turán graph, $T_2(n)$, with a graph $F$ inserted into one of the parts. Now if $F$ does not contain a 4-cycle, then $G$ is $H$-free. Furthermore, $\chi(G) \geqslant \chi(F)$ and $\delta(G) \geqslant \lfloor n/2 \rfloor$. Thus, taking $F$ to have girth at least five and arbitrarily large chromatic number shows that $\delta_\chi(H) \geqslant 1/2$. But, of course, all $H$-free graphs have at most $(1/2 + o(1))\binom{n}{2}$ edges and so minimum degree at most $(1/2 + o(1))n$. In particular,

$$1/2 = \delta_\chi(H) = \delta_\chi(H, k),$$

for all positive integers $k$. This is unsatisfying, failing to capture the macroscopic behaviour of $H$-free graphs with large minimum degree. All these graphs are close to (within $o(n^2)$ edges of) being bipartite – they are $T_2(n)$ with a smattering more edges – and their high chromatic number is rather artificial. For these reasons, a natural (and possibly better) notion of the structure of $H$-free graphs with large minimum degree is the *approximate chromatic profile*. This is

$$\begin{aligned}\delta^*_\chi(H, k) = \inf\{c \colon \text{if } |G| = n,\ \delta(G) \geqslant cn, \text{ and } G \text{ is } H\text{-free},\\ \text{then } G \text{ can be made } k\text{-colourable by deleting } o(n^2) \text{ edges}\}.\end{aligned}$$

Note in passing that $\delta_\chi(H, k) \geqslant \delta^*_\chi(H, k)$, although equality need not occur. Indeed, we have just seen that the chromatic profile of $H = K_3(2)$ is the constant $1/2$ sequence, while Theorem 5.5 below shows that $H$'s approximate chromatic profile is the same as the chromatic profile of triangle-free graphs.

If $H$ is $(r + 1)$-chromatic, then the $r$-partite Turán graph, $T_r(n)$, is $H$-free and cannot be made $(r - 1)$-colourable without deleting $\Omega(n^2)$ edges. Also, any $n$-vertex $H$-free graph has at most $(1 - 1/r + o(1))\binom{n}{2}$ edges and so has minimum degree at most $(1 - 1/r + o(1))n$. Thus, for all $k \leqslant r - 1$ we have $\delta^*_\chi(H, k) = 1 - 1/r$. In particular, the first interesting threshold in the approximate chromatic profile of $H$ is

$$\delta^*_\chi(H, \chi(H) - 1),$$

which is exactly $\delta_H$.

For a family of graphs $\mathcal{F}$, one can make the more general definition

$$\delta^*_\chi(\mathcal{F},k) = \inf\{c\colon \text{if } |G| = n,\ \delta(G) \geqslant cn, \text{ and } G \in \mathcal{F}, \\ \text{then } G \text{ can be made } k\text{-colourable by deleting } o(n^2) \text{ edges}\}.$$

This again satisfies the inequality $\delta_\chi(\mathcal{F},k) \geqslant \delta^*_\chi(\mathcal{F},k)$. When $\mathcal{F}$ is closed under blow-ups, we have equality (this is proved in § 5.5).

**Theorem 5.4.** *Let $\mathcal{F}$ be a family of graphs that is closed under taking blow-ups. For any positive integer $k$,*

$$\delta^*_\chi(\mathcal{F},k) = \delta_\chi(\mathcal{F},k).$$

Note that the families of triangle-free graphs, of $K_{r+1}$-free graphs, and of $a$-locally $b$-partite graphs are closed under taking blow-ups. Theorem 5.4 explains why the Andrásfai-Erdős-Sós theorem is such a clean stability theorem, not requiring the deletion of any edges.

For non-complete $H$, the family of $H$-free graphs is not closed under taking blow-ups (if $H$ is $(r+1)$-chromatic, then $K_{r+1}$ is $H$-free, while not all of its blow-ups are). However, there is a natural family of graphs, which is closed under taking blow-ups, and whose chromatic profile is the same as the approximate chromatic profile of $H$. Define *$H$-hom* to be the family of *$H$-homomorphism-free graphs*, that is, those graphs to which $H$ is not homomorphic. This is closed under taking blow-ups (see the discussion at the end of § 1.3). Then

$$\delta_\chi(H\text{-hom},k) = \inf\{c\colon \text{if } \delta(G) \geqslant c|G| \text{ and } H \text{ is not homomorphic to } G, \\ \text{then } G \text{ is } k\text{-colourable}\}$$

is the chromatic profile of this family. In § 5.5, we will show that this is identical to the approximate chromatic profile of $H$ and so $\delta_H = \delta_\chi(H\text{-hom},\chi(H)-1)$. This further promotes what natural notions $\delta_H$ and the approximate chromatic profile of $H$ are.

**Theorem 5.5.** *For any graph $H$ and any positive integer $k$,*

$$\delta^*_\chi(H,k) = \delta_\chi(H\text{-hom},k).$$

### 5.1.2 The Graphs in Theorem 5.3 and some Motivation

Various graphs appear as $F_g$ in the statement of Theorem 5.3. Here we define them explicitly and provide some motivation for their presence. The $F_g$ and $c_g$ are given in Table 5.1. Notice that the $F_g$ all appeared in Figure 3.1 and were discussed thoroughly in § 3.1.1. We show them again in Figure 5.1 for convenience.

| $g$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $F_g$ | $W_5$ | $W_7$ | $\overline{C}_7$ | $W_9$ | $H_2^+$ | $W_{11}$ | $H_2$ | $W_{13}$ | $T_0$ | $W_{15}$ | $H_1^{++}$ |
| $c_g$ | $\frac{2}{3}$ | $\frac{2}{5}$ | $\frac{1}{3}$ | $\frac{2}{7}$ | $\frac{1}{4}$ | $\frac{2}{9}$ | $\frac{1}{5}$ | $\frac{2}{11}$ | $\frac{1}{6}$ | $\frac{2}{13}$ | $\frac{1}{7}$ |

Table 5.1: $F_g$ and $c_g$

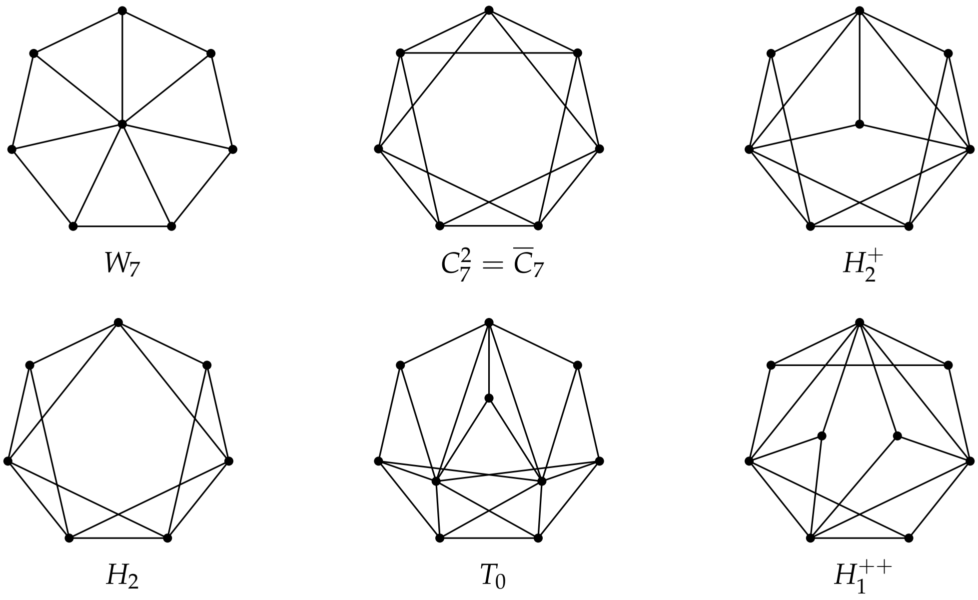



Figure 5.1: $F_g$

The sequence $F_g$ is slightly unusual. Firstly the graphs do not increase in size. Secondly it is not always true that $F_{g+1}$ is homomorphic to $F_g$ and so it is, for example, possible for a graph to be homomorphic to both $F_3$ and $F_5$ but not $F_4$.

We now motivate why it is these graphs that are the $F_g$. The intuitive explanation for Theorem 5.2 is that the main obstacle for being close to (that is, within $o(n^2)$ edges of) bipartite is containing some blow-up of an odd cycle. That odd cycle must be consistent with being $H$-free (in particular, the blow-up of the odd cycle must be $H$-free) and hence it is the first odd cycle to which $H$ is not homomorphic that determines $\delta_H$.

Now consider the $r = 3$ version of Theorem 5.3: we are interested in which graphs' blowups are the main obstacles for being close to tripartite. Given the importance of odd cycles for being far from bipartite it seems natural that odd wheels would be obstacles here and indeed six of the $F_g$ are odd wheels. Other $F_g$ do not contain any odd wheels and so are locally bipartite. These observations suggest we should pay attention to 4-chromatic locally bipartite graphs as these may be obstacles for being close to tripartite. More generally, we should pay attention to $(a+3)$-chromatic $a$-locally bipartite graphs as obstacles for being close to $(a+2)$-colourable (together with $a$-cliques joined to odd wheels). We did this in § 4.4.2 and the theorem obtained there, Theorem 4.20, which we restate here, will be used as part of our proof of Theorem 5.3.

**Theorem 4.20** ($a$-locally bipartite graphs)**.** *Let $G$ be an $a$-locally bipartite graph.*

- *If $\delta(G) > (1 - 1/(a + 4/3)) \cdot |G|$, then $G$ is $(a + 2)$-colourable. Blow-ups of $K_{a-1} + \overline{C}_7$ show that this is tight.*
- *If $\delta(G) > (1 - 1/(a + 5/4)) \cdot |G|$, then $G$ is either $(a + 2)$-colourable or contains $K_{a-1} + \overline{C}_7$.*
- *If $\delta(G) > (1 - 1/(a + 6/5)) \cdot |G|$, then $G$ is either $(a + 2)$-colourable or contains $K_{a-1} + \overline{C}_7$ or $K_{a-1} + H_2^+$.*
- *If $\delta(G) > (1 - 1/(a + 7/6)) \cdot |G|$, then $G$ is either $(a + 2)$-colourable or contains $K_{a-1} + H_2$.*
- *If $\delta(G) > (1 - 1/(a + 8/7)) \cdot |G|$, then $G$ is either $(a + 2)$-colourable or contains $K_{a-1} + H_2$ or $K_{a-1} + T_0$.*

One might ask whether there are other sequences $F'_g$ and $c'_g$ for which Theorem 5.3 holds. The fact that $\delta_H$ is a fixed number has two corollaries. Firstly, it must be the case that $c'_g = c_g$ for $1 \leqslant g \leqslant 11$. Secondly, any graph homomorphic to all of $F_1, F_2, \ldots, F_g$ must also be homomorphic to all of $F_1, F_2, \ldots, F_{g-1}, F'_g$ and vice versa. It seems likely that the $F_g$ are the minimal graphs satisfying this and so form the "canonical" sequence, but I have been unable to prove this.

### 5.1.3 Tools

We will make great use of Szemerédi's regularity lemma [Sze78] together with some associated machinery which we describe here. Let $(X, Y)$ be a pair of vertex subsets of graph $G$. We use $d(X, Y) = e(X, Y)|X|^{-1}|Y|^{-1}$ to denote the *density* between $X$ and $Y$. The pair $(X, Y)$ is *ε-regular* if

$$|d(U, V) - d(X, Y)| \leqslant \varepsilon$$

for all $U \subset X$, $V \subset Y$ with $|U| \geqslant \varepsilon|X|$ and $|V| \geqslant \varepsilon|Y|$. A partition $\mathcal{P} = V_0 \cup V_1 \cup \cdots \cup V_k$ of $V(G)$ is an *ε-regular partition* if:

- $V_1, V_2, \ldots, V_k$ all have equal size and $|V_0| \leqslant \varepsilon|G|$.
- For all but at most $\varepsilon\binom{k}{2}$ pairs $ij$ ($i < j$), the pair $(V_i, V_j)$ is $\varepsilon$-regular.

Szemerédi's celebrated result is that, for every positive integer $\ell$ and $\varepsilon > 0$, there is some $L = L(\ell, \varepsilon)$ such that every graph with at least $\ell$ vertices has an $\varepsilon$-regular partition into at least $\ell$ but at most $L$ parts. We will need a version of Szemerédi's regularity lemma which works well with minimum degrees.

Fix $\varepsilon > 0$ and some $\lambda \geqslant 0$. Suppose we have a graph $G$ with some $\varepsilon$-regular partition $\mathcal{P} = V_0 \cup V_1 \cup \cdots \cup V_k$. These induce what is called a *reduced graph* $R(\mathcal{P}, \varepsilon, \lambda)$: this has vertex set $[k] = \{1, 2, \ldots, k\}$ with vertex $i$ joined to vertex $j$ exactly if the pair $(V_i, V_j)$ is $\varepsilon$-regular and $d(V_i, V_j) \geqslant \lambda$. Note when $\lambda = 0$ this graph will have at least $(1-\varepsilon)\binom{k}{2}$ edges by the definition of an $\varepsilon$-regular partition. We will make use of the following version of the regularity lemma which is an immediate corollary of Theorem 1.10 in Komlós and Simonovits's survey of the subject [KS96].

**Lemma 5.6** (Szemerédi's regularity lemma, minimum degree form)**.** *Let $\varepsilon > 0$ and $\ell$ be a positive integer. There is a positive integer $L$ such that the following holds for all $n \geqslant \ell$ and $\lambda, \delta \in [0, 1]$ . If $G$ is graph on $n$ vertices with $\delta(G) \geqslant \delta n$, then $G$ has some $\varepsilon$-regular partition $\mathcal{P} = V_0 \cup V_1 \cup \cdots \cup V_k$ with $k$ between $\ell$ and $L$ such that the corresponding reduced graph $R(\mathcal{P}, \varepsilon, \lambda)$ has minimum degree at least $(\delta - \varepsilon - \lambda)k$.*

The point of the reduced graph is that if it contains some structure, then we can find a large structure in $G$, by using the following building lemma (for example, see [KS96, Thm 2.1]).

**Lemma 5.7** (graph-building lemma)**.** *Let $H$ be a graph (on vertex set $\{1, 2, \ldots, |H|\}$), $t$ a positive integer, and $\lambda \in (0, 1)$. For all sufficiently small $\varepsilon > 0$ the following holds. Suppose $V_1$, $\ldots$, $V_{|H|}$ are sufficiently large pairwise disjoint vertex sets with $(V_i, V_j)$ $\varepsilon$-regular of density at least $\lambda$ for each $ij \in E(H)$. Then we can find a copy of $H(t)$ with each blown-up vertex in the corresponding $V_i$.*

These two lemmas work well together: given a large graph $G$ with minimum degree $\delta|G|$, Lemma 5.6 shows that $G$ has a corresponding reduced graph $R$ of bounded size and with minimum degree almost $\delta|R|$. If $G$ is $H(t)$-free, then, by Lemma 5.7, $R$ is $H$-free. This may give some structural information about $R$ (e.g. it is $r$-colourable) which we can then pull back to $G$.

Finally, odd cycles will play an important role in determining $\delta_H$, so we note the following fact about homomorphisms to odd cycles.

**Lemma 5.8.** *Let $g$ be a positive integer. If $G$ is homomorphic to $C_{2g+1}$, then $G$ contains no odd cycles of length less than $2g + 1$.*

*Proof.* Let $\varphi$ be a homomorphism from $G$ to $C_{2g+1}$. Let $C$ be an odd cycle of $G$. The restriction of $\varphi$ to $C$ gives a homomorphism from $C$ to $\varphi(C)$, so $\chi(\varphi(C)) \geqslant \chi(C) = 3$. Thus, $\varphi(C)$ is the whole of $C_{2g+1}$, so $|C| \geqslant |\varphi(C)| = 2g + 1$. $\square$

## 5.2 $\delta_H$ for 3-Chromatic $H$ – Proof of Theorem 5.2

Let $H$ be a graph with chromatic number three, so $H$ is homomorphic to $K_3 \cong C_3$. Furthermore, $H$ is not bipartite, so contains at least one odd cycle. This, coupled with Lemma 5.8, means that $H$ is homomorphic to only finitely many odd cycles. Let $g$ be the smallest positive integer with $H$ not homomorphic to $C_{2g+1}$. A balanced blow-up of $C_{2g+1}$ on $n$ vertices is $H$-free and has minimum degree at least $2\lfloor n/(2g+1)\rfloor$. Further, by Lemma 1.7, this balanced blow-up requires the deletion of $\Omega_g(n^2)$ edges to be made bipartite. In particular,

$$\delta_H \geqslant \frac{2}{2g+1}.$$

We claim that in fact there is equality. Before proving this, we need the following result for odd cycles, which was noted by Andrásfai, Erdős, and Sós [AES74]. For completeness we give a proof.

**Lemma 5.9.** *Let $g \geqslant 2$ be a positive integer. Suppose $G$ is a non-bipartite graph with*

$$\delta(G) > \frac{2}{2g+1} \cdot |G|.$$

*Then $G$ contains an odd cycle of length less than $2g+1$.*

*Proof.* Let $C$ be the shortest odd cycle in $G$. By minimality, $C$ is induced and no vertex is adjacent to three vertices in $C$. Thus,

$$|C| \cdot \delta(G) \leqslant e(C, G) \leqslant 2|G|,$$

and so $|C| \leqslant 2|G|/\delta(G) < 2g+1$. □

We are now ready to prove Theorem 5.2 which determines $\delta_H$ for 3-chromatic $H$.

**Theorem 5.2** ($\delta_H$ for 3-chromatic $H$)**.** *Let $H$ be a 3-chromatic graph. There is a smallest positive integer $g$ such that $H$ is not homomorphic to $C_{2g+1}$. Then*

$$\delta_H = \frac{2}{2g+1}.$$

*Proof.* The graph $H$ is not bipartite so contains an odd cycle. In particular, if $g$ is such that $2g+1$ is greater than the length of the shortest odd cycle of $H$, then, by Lemma 5.8, $H$ is not homomorphic to $C_{2g+1}$. Take $g$ minimal with $H$ not homomorphic to $C_{2g+1}$.

By the opening remarks of this section, a balanced blow-up of $C_{2g+1}$ on $n$ vertices is

$H$-free, has minimum degree at least $2\lfloor n/(2g+1)\rfloor$ and requires the deletion of at least $\Omega_g(n^2)$ edges to be made bipartite so $\delta_H \geqslant 2/(2g+1)$.

We are left to show that, for all $\eta > 0$, any $n$-vertex $H$-free graph with minimum degree at least
$$\left(\frac{2}{2g+1}+\eta\right)n,$$
can be made bipartite by deleting at most $\eta n^2$ edges when $n$ is sufficiently large. Firstly, note that $H$ is homomorphic to all of $C_3$, $C_5$, $\dots$, $C_{2g-1}$, so there exists some positive integer $t$ such that $H$ is a subgraph of all of $C_3(t)$, $C_5(t)$, $\dots$, $C_{2g-1}(t)$.

Fix $n$ and $\ell$ large (chosen later), let $\lambda = \eta/2$ and $\varepsilon > 0$ be sufficiently small. Let $G$ be a $n$-vertex graph with minimum degree at least $(2/(2g+1)+\eta)n$. By Lemma 5.6, $G$ has some $\varepsilon$-regular partition $\mathcal{P} = V_0 \cup V_1 \cup \cdots \cup V_k$ with $k$ between $\ell$ and $L$ (a constant not depending on $G$ or $n$) such that the reduced graph $R = R(\mathcal{P}, \varepsilon, \lambda)$ has minimum degree greater than $2k/(2g+1)$. By Lemma 5.9, $R$ is either bipartite or contains one of $C_3$, $C_5$, $\dots$, $C_{2g-1}$.

Applying Lemma 5.7, provided $\varepsilon$ was chosen small enough (in terms of $\lambda$) and $n$ is large enough, either $R$ is bipartite or $G$ contains one of $C_3(t)$, $C_5(t)$, $\dots$, $C_{2g-1}(t)$. The latter contradicts $G$ being $H$-free and so $R$ is bipartite.

Now, consider deleting from $G$ all edges within each $V_i$, the edges incident to $V_0$, and all edges between $V_i$ and $V_j$ when $ij \notin E(R)$. The resulting graph is a blow-up of $R$, so is bipartite. This process deletes at most
$$\begin{aligned}
&k \cdot (n/k)^2 + \varepsilon n \cdot n + \varepsilon\tbinom{k}{2}(n/k)^2 + \lambda n^2 \\
&\leqslant n^2(\lambda + 2\varepsilon + 1/k) \\
&\leqslant n^2(\eta/2 + 2\varepsilon + 1/\ell) \leqslant \eta n^2
\end{aligned}$$
edges, provided that $\ell$ is large enough and $\varepsilon$ is small enough. □

**Remark 5.10.** By Lemma 5.8, the odd girth of $G$ is at least $2g-1$ (where $g$ is as in the theorem statement). However, we may not have equality. For example, the Petersen graph has odd girth 5 but is not homomorphic to $C_5$, so $\delta_{\text{Petersen}}$ is $2/5$ and not $2/7$.

## 5.3 Properties of $\delta_H$

We take a moment to crystallise the key ingredients of the proof of Theorem 5.2 (and, in particular, what role the odd cycles played). Fix $r \geqslant 2$ and suppose we have a sequence of graphs $L_0 = K_{r+1}$, $L_1$, $L_2$, $\dots$, $L_m$ and a sequence of constants $k_1$, $k_2$, $\dots$, $k_m$ where

$m$ may be infinity (if both sequences are infinite). The relevant properties this pair of sequences might satisfy are:

1. None of $L_1, L_2, L_3, \dots$ is $r$-colourable.
2. No $(r+1)$-chromatic graph is homomorphic to all of $L_1, L_2, \dots$.
3. For each $g$, if $G$ is an $n$-vertex graph with minimum degree greater than $k_g n$, then $G$ is either $r$-colourable or contains at least one of $L_0, L_1, \dots, L_{g-1}$.
4. For each $g$ and any $c < k_g$, there is some blow-up $L'_g$ of $L_g$ satisfying $\delta(L'_g) \geqslant c \cdot |L'_g|$.

Theorem 5.2 corresponds to the sequences $L_g = C_{2g+3}$ and $k_g = 2/(2g+3)$ satisfying all the properties for $r = 2$. As odd cycles are regular, we did not make use of the blow-ups in Property 4. However, the $L_g$ we use later will often be non-regular and furthermore they may have no blow-ups with $\delta(L'_g) = k_g \cdot |L'_g|$, but still satisfy Property 4. One could weaken Property 3 to minimum degree greater than $(k_g + o(1))n$ (which is all we use in our analysis below) but for our purposes this is unnecessary.

Let $H$ be an $(r+1)$-chromatic graph. Suppose $H$ is not homomorphic to $L_g$ and Properties 1 and 4 hold. Then, for any $c < k_g$, let $L'_g$ be a blow-up of $L_g$ with $\delta(L'_g) \geqslant c \cdot |L'_g|$ and let $G$ be a balanced blow-up of $L'_g$ on $n$ vertices. As $H$ is not homomorphic to $L_g$, $G$ is $H$-free. Furthermore, $G$ has minimum degree at least $\delta(L'_g)\lfloor n/|L'_g|\rfloor \geqslant (c - o(1))n$. By Property 1, $L'_g$ is not $r$-colourable and so, by Lemma 1.7, to make $G$ $r$-colourable requires the deletion of $\Omega(n^2)$ edges. Thus $\delta_H \geqslant c$ and so $\delta_H \geqslant k_g$.

Suppose $H$ is homomorphic to all of $L_1, L_2, \dots, L_{g-1}$ and Property 3 holds (note that $H$ is also homomorphic to $L_0$). Then the same regularity argument as in the proof of Theorem 5.2 shows that $\delta_H \leqslant k_g$. We now sketch this argument. Let $\eta > 0$ and take a large graph $G$ with minimum degree at least $(k_g + \eta)|G|$. We can use Lemma 5.6 to get a corresponding reduced graph $R$ with minimum degree greater than $(k_g + \eta')|R|$ (some $\eta' \in (0, \eta)$), which, by Property 3, is either $r$-colourable or contains one of $L_0, L_1, \dots, L_{g-1}$. In the latter case, we use Lemma 5.7 to get a copy of $H$ in $G$ and in the former case we may delete at most $\eta|G|^2$ edges from $G$ to leave an $r$-colourable graph.

The upshot of all this is that if Properties 1, 3 and 4 hold, then either $\delta_H \leqslant k_m$ (if $H$ is homomorphic to all of $L_1, L_2, \dots, L_m$) or $\delta_H = k_g$ where $g$ is minimal with $H$ not homomorphic to $L_g$. Of course, if Property 2 also holds, then we can determine $\delta_H$ for any $(r+1)$-chromatic $H$. Even without Property 2, any sequence does determine $\delta_H$ for many $H$ and gives an upper bound for the rest. Thus, we are particularly interested in pairs of sequences satisfying Properties 1, 3 and 4.

We illustrate these remarks by next proving a weak version of Theorem 5.3. Note that, when $r = 2$, $\delta_H$ could be arbitrarily close to zero (corresponding to $k_g \to 0$ as $g \to \infty$).

However, this is not the case for larger $r$.

**Theorem 5.11.** *If $H$ is a graph with chromatic number $r+1 \geqslant 3$, then there is a least $g$ such that $H$ is not homomorphic to $K_{r-2} + C_{2g+1}$ and furthermore*

$$1 - \frac{1}{r-1} < 1 - \frac{1}{r-1+2/(2g-1)} \leqslant \delta_H \leqslant 1 - \frac{1}{r-1/3}.$$

*Proof.* The right-hand inequality is just inequality (5.1). For the middle inequality define the following sequence of graphs: $L_g = K_{r-2} + C_{2g+3}$ (so $L_0 = K_{r+1}$) and let

$$k_g = 1 - \frac{1}{r-1+2/(2g+1)}.$$

From the preceding discussion it suffices to show that these sequences satisfy Properties 1, 2 and 4 (but not necessarily 3). Indeed, if these properties hold, then there is a minimal $g$ such that $H$ is not homomorphic to $L_{g-1}$ and so

$$\delta_H \geqslant k_{g-1} = 1 - \frac{1}{r-1+2/(2g-1)} > 1 - \frac{1}{r-1}.$$

Property 1 is immediate: each $L_g$ has chromatic number $\chi(K_{r-2}) + \chi(C_{2g+3}) = r-2+3 = r+1$.

Suppose $H$ is some $(r+1)$-chromatic graph which is homomorphic to all the $L_g$ and let the homomorphisms be $\varphi_g \colon H \to L_g = K_{r-2} + C_{2g+3}$. Now $\varphi_g^{-1}(K_{r-2})$ is $(r-2)$-colourable, so $X_g = \varphi_g^{-1}(C_{2g+3})$ is not bipartite (as $H$ is not $r$-colourable). In particular, for each $g$, $H[X_g]$ is not bipartite but $H[X_g] \to C_{2g+3}$. Thus, by Lemma 5.8, $H[X_g]$ contains an odd cycle of length at least $2g+3$. Therefore, $H$ contains odd cycles of arbitrary length, which is absurd. Hence, we have Property 2.

Finally, for Property 4, let $L'_g = K_{r-2}(2g+1) + C_{2g+3}$, which is a blow-up of $L_g$. The graph $L'_g$ has $(2g+1)(r-2)+2g+3 = (2g+1)(r-1)+2$ vertices and is $[(2g+1)(r-2)+2]$-regular. In particular,

$$k_g \cdot |L'_g| = \left(1 - \frac{2g+1}{(r-1)(2g+1)+2}\right) \cdot [(2g+1)(r-1)+2] = \delta(L'_g). \qquad \square$$

## 5.4 $\delta_H$ for General $H$ – Proof of Theorem 5.3

We are now ready to prove Theorem 5.3, which we restate here with the explicit $F_g$ and $c_g$ for convenience.

**Theorem 5.12.** *Define the sequence of graphs $F_g$ and constants $c_g$ $(1 \leqslant g \leqslant 11)$ as follows.*

| $g$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $F_g$ | $W_5$ | $W_7$ | $\overline{C}_7$ | $W_9$ | $H_2^+$ | $W_{11}$ | $H_2$ | $W_{13}$ | $T_0$ | $W_{15}$ | $H_1^{++}$ |
| $c_g$ | $\frac{2}{3}$ | $\frac{2}{5}$ | $\frac{1}{3}$ | $\frac{2}{7}$ | $\frac{1}{4}$ | $\frac{2}{9}$ | $\frac{1}{5}$ | $\frac{2}{11}$ | $\frac{1}{6}$ | $\frac{2}{13}$ | $\frac{1}{7}$ |

*Let $r \geqslant 3$ be an integer. If $H$ is a $(r+1)$-chromatic graph and $g$ is minimal such that $H$ is not homomorphic to $K_{r-3} + F_g$, then*

$$\delta_H = 1 - \frac{1}{r - 1 + c_g}.$$

*If $H$ is homomorphic to all eleven $K_{r-3} + F_g$, then there is a least $g$ for which $H$ is not homomorphic to $K_{r-2} + C_{2g+1}$: $\delta_H$ satisfies the bounds*

$$1 - \frac{1}{r-1} < 1 - \frac{1}{r - 1 + 2/(2g-1)} \leqslant \delta_H \leqslant 1 - \frac{1}{r - 1 + 1/7}.$$

*Proof.* The lower bound

$$1 - \frac{1}{r - 1 + 2/(2g-1)} \leqslant \delta_H$$

in the final part of this theorem follows from Theorem 5.11. The discussion following Properties 1 to 4 shows that what remains is to check that the sequences

$$L_g = K_{r-3} + F_g, \qquad k_g = 1 - \frac{1}{r - 1 + c_g}$$

satisfy Properties 1, 3 and 4. Property 1 is immediate: for each $g$, $\chi(L_g) = r - 3 + \chi(F_g) = r + 1$.

We now consider Property 4. When $F_g = W_{2k+1}$, $c_g = 2/(2k-1)$. For $L_g = K_{r-3} + W_{2k+1} = K_{r-2} + C_{2k+1}$, we take $L'_g = K_{r-2}(2k-1) + C_{2k+1}$. This is a regular blow-up of $L_g$ and, as was shown in the proof of Theorem 5.11, satisfies

$$\left(1 - \frac{1}{r - 1 + c_g}\right) \cdot |L'_g| = \delta(L'_g),$$

which is exactly $k_g \cdot |L'_g| = \delta(L'_g)$. When $F_g = \overline{C}_7$ (i.e. $g = 3$), $c_g = 1/3$. We take $L'_g = K_{r-3}(3) + \overline{C}_7$ which is a regular blow-up of $L_g$. This satisfies

$$\delta(L'_g)|L'_g|^{-1} = \frac{3r-5}{3r-2} = 1 - \frac{1}{r - 1 + 1/3} = k_g.$$

This establishes Property 4 except when $g$ is $5, 7, 9$, or $11$. Figure 3.3 on page 33 shows weightings (which induce blow-ups) of the $F_g$ which maximise the minimum degree relative to the order. However, some care will be needed owing to the zero weights. As discussed in § 3.1.1, while there may be no blow-up of $F_5 = H_2^+$ whose minimum degree divided by order is $5/9$, for any $k < 5/9$, there is a genuine blow-up $H_2^{+\prime}$ of $H_2^+$ with $\delta(H_2^{+\prime}) \geqslant k \cdot |H_2^{+\prime}|$.

Let $c < k_5$: $c = 1 - 1/(r-1+\beta)$ for some $\beta < c_5 = 1/4$ and so $k = 1 - 1/(2+\beta) < 5/9$. Let $H_2^{+\prime}$ be a blow-up of $H_2^+$ satisfying $\delta(H_2^{+\prime}) \geqslant k \cdot |H_2^{+\prime}|$. Let $L_5' = K_{r-3}(|H_2^{+\prime}| - \delta(H_2^{+\prime})) + H_2^{+\prime}$ which is a blow-up of $L_5$. Then

$$\begin{aligned}
\delta(L_5')|L_5'|^{-1} &= \frac{(r-3)|H_2^{+\prime}| - (r-4)\delta(H_2^{+\prime})}{(r-2)|H_2^{+\prime}| - (r-3)\delta(H_2^{+\prime})} \\
&= 1 - \frac{|H_2^{+\prime}| - \delta(H_2^{+\prime})}{(r-2)|H_2^{+\prime}| - (r-3)\delta(H_2^{+\prime})} \\
&= 1 - \frac{1}{r-3+1/(1-\delta(H_2^{+\prime})|H_2^{+\prime}|^{-1})} \\
&\geqslant 1 - \frac{1}{r-3+1/(1-k)} \\
&= 1 - \frac{1}{r-3+2+\beta} = c,
\end{aligned}$$

which establishes Property 4 for $g = 5$. Similar calculations hold for $g = 7, 9, 11$ (for $g = 7$, there are even no troublesome zero weights). Indeed, for $c < k_g$: $c = 1 - 1/(r-1+\beta)$ for some $\beta < c_g$ and so $k = 1 - 1/(2+\beta)$ is less than $6/11$ (for $g = 7$), less than $7/13$ (for $g = 9$) and less than $8/15$ (for $g = 11$). Figure 3.3 shows that there is a blow up $F_g'$ of $F_g$ with $\delta(F_g') \geqslant k \cdot |F_g'|$. Then $L_g' = K_{r-3}(|F_g'| - \delta(F_g')) + F_g'$ is a blow-up of $L_g$, which satisfies $\delta(L_g')|L_g'|^{-1} \geqslant c$. Thus the sequences satisfy Property 4.

We are left to show that the sequences satisfy Property 3. We will make use of the results about $a$-locally bipartite graphs given in Theorem 4.20. Let $G$ be a graph with minimum degree greater than $k_g \cdot |G|$.

First suppose that $G$ is not $(r-2)$-locally bipartite. Then there is an $(r-2)$-clique $K$ in $G$ whose common neighbourhood is not bipartite. Let $G_K$ be the induced subgraph of $G$ whose vertex set is the common neighbourhood of the vertices in $K$. Let the vertices of $K$ be $x_1, x_2, \ldots, x_{r-2}$. Note that for each $v \in V(G)$,

$$\mathbb{1}(v \in G_K) \geqslant \mathbb{1}(vx_1 \in E(G)) + \cdots + \mathbb{1}(vx_{r-2} \in E(G)) - (r-3)$$

and so summing over all the vertices gives

$$|G_K| \geqslant (r-2)\delta(G) - (r-3)|G| > \big((r-2)k_g - (r-3)\big) \cdot |G|.$$

Note that $\delta(G_K) \geqslant \delta(G) - (|G| - |G_K|) = |G_K| - (|G| - \delta(G))$ so

$$\begin{aligned}\frac{\delta(G_K)}{|G_K|} &\geqslant 1 - \frac{|G| - \delta(G)}{|G|} \cdot \frac{|G|}{|G_K|} > 1 - \left(1 - \frac{\delta(G)}{|G|}\right) \cdot \frac{1}{(r-2)k_g - (r-3)} \\ &> 1 - \frac{1 - k_g}{(r-2)k_g - (r-3)} = 1 - \frac{1}{1 + c_g}.\end{aligned}$$

For $g = 1$: $G_K$ is not bipartite and $\delta(G_K) > 2/5 \cdot |G_K|$ so, by Lemma 5.9, $G_K$ contains a triangle and so $G$ contains a $K_{r-2} + K_3 = L_0$. Similarly for $g = 2$, $G$ contains a copy of $L_0$ or $L_1$ and for $g = 3$, $G$ contains one of $L_0$, $L_1$ and $L_2$. More generally, for $g$ even, $G$ contains $L_\ell$ for some $\ell \in \{0, 1, 2, 4, 6, \ldots, g-2\}$ and for $g$ odd, $G$ contains $L_\ell$ for some $\ell \in \{0, 1, 2, 4, 6, \ldots, g-1\}$.

Now suppose that $G$ is $(r-2)$-locally bipartite. For $g = 1, 2, 3$:

$$\delta(G)|G|^{-1} > 1 - \frac{1}{r - 1 + c_3} = 1 - \frac{1}{r - 2 + 4/3},$$

so, by Theorem 4.20, $G$ is $r$-colourable. For $g = 4, 5$:

$$\delta(G)|G|^{-1} > 1 - \frac{1}{r - 1 + c_5} = 1 - \frac{1}{r - 2 + 5/4},$$

so, by Theorem 4.20, $G$ is $r$-colourable or contains $K_{r-3} + \overline{C}_7 = L_3$. For $g = 6, 7$:

$$\delta(G)|G|^{-1} > 1 - \frac{1}{r - 1 + c_7} = 1 - \frac{1}{r - 2 + 6/5},$$

so, by Theorem 4.20, $G$ is $r$-colourable or contains $K_{r-3} + \overline{C}_7 = L_3$ or contains $K_{r-3} + H_2^+ = L_5$. For $g = 8, 9$:

$$\delta(G)|G|^{-1} > 1 - \frac{1}{r - 1 + c_9} = 1 - \frac{1}{r - 2 + 7/6},$$

so, by Theorem 4.20, $G$ is $r$-colourable or contains $K_{r-3} + H_2 = L_7$. Finally, for $g = 10, 11$:

$$\delta(G)|G|^{-1} > 1 - \frac{1}{r - 1 + c_{11}} = 1 - \frac{1}{r - 2 + 8/7},$$

so, by Theorem 4.20, $G$ is $r$-colourable or contains $K_{r-3} + H_2 = L_7$ or contains $K_{r-3} + T_0 = L_9$. Hence, the sequences do indeed satisfy Property 3, as required. □

## 5.5 Homomorphism-free Graphs and Approximate Chromatic Profiles

In § 5.1.1, we introduced two more chromatic profiles related to $\delta_H$. The first was the approximate chromatic profile which for a family $\mathcal{F}$ is,

$$\delta^*_\chi(\mathcal{F},k) = \inf\{c\colon \text{if } |G| = n, \delta(G) \geqslant cn, G \in \mathcal{F}, \\ \text{then } G \text{ can be made } k\text{-colourable by deleting at most } o(n^2) \text{ edges}\},$$

and the second was the chromatic profile of the family of $H$-homomorphism-free graphs,

$$\delta_\chi(H\text{-hom},k) = \inf\{c\colon \text{if } \delta(G) \geqslant c|G| \text{ and } H \nrightarrow G, \text{then } G \text{ is } k\text{-colourable}\},$$

We first prove Theorem 5.4, showing that families which are closed under blow-ups have well-behaved chromatic profiles.

**Theorem 5.4.** *Let $\mathcal{F}$ be a family of graphs that is closed under taking blow-ups. For any positive integer $k$,*

$$\delta^*_\chi(\mathcal{F},k) = \delta_\chi(\mathcal{F},k).$$

*Proof.* Plainly the definitions of the approximate and normal chromatic profiles give

$$\delta_\chi(\mathcal{F},k) \geqslant \delta^*_\chi(\mathcal{F},k),$$

as being $k$-colourable implies being within $o(n^2)$ edges of $k$-colourable.

Let $c < \delta_\chi(\mathcal{F},k)$. By definition, there must be a graph $G \in \mathcal{F}$ which is not $k$-colourable and has minimum degree at least $c|G|$. Let $G'$ be the balanced blow-up of $G$ on $n$ vertices: as $\mathcal{F}$ is closed under blow-ups, $G' \in \mathcal{F}$. Furthermore, $G'$ has minimum degree at least $c|G|\lfloor n/|G|\rfloor = (c + o(1))n$. Finally, Lemma 1.7 shows that making $G'$ $k$-colourable requires the deletion of $\Omega(n^2)$ edges. In particular,

$$\delta^*_\chi(\mathcal{F},k) \geqslant c,$$

as required. □

Finally, we prove Theorem 5.5 showing that the chromatic profile of $H$-homomorphism-free graphs and the approximate chromatic profile of $H$-free graphs are one and the same.

**Theorem 5.5.** *For any graph $H$ and any positive integer $k$,*

$$\delta^*_\chi(H,k) = \delta_\chi(H\text{-hom},k).$$

*Proof.* We first show that $\delta^*_\chi(H,k) \geqslant \delta_\chi(H\text{-hom},k)$. Let $c < \delta_\chi(H\text{-hom},k)$, so there is a non-$k$-colourable graph $G$ with minimum degree at least $c|G|$ and to which $H$ is not homomorphic. Let $G'$ be a balanced blow-up of $G$ on $n$ vertices: $H$ is not homomorphic to $G$, so $G'$ is $H$-free. Furthermore, $G'$ has minimum degree at least $c|G|\lfloor n/|G|\rfloor$. Finally, Lemma 1.7 shows that making $G'$ $k$-colourable requires the deletion of $\Omega(n^2)$ edges. In particular, $\delta^*_\chi(H,k) \geqslant c$, as required.

We now show that $\delta^*_\chi(H,k) \leqslant \delta_\chi(H\text{-hom},k)$. Let $\gamma > 0$, $c = \delta_\chi(H\text{-hom},k) + 2\gamma$ and let $G$ be an $n$-vertex $H$-free graph with $\delta(G) \geqslant cn$. Let

$$\mathcal{H} = \{H' \colon H \to H', |H'| \leqslant |H|\},$$

and note that this is finite and that $H$ is homomorphic to a graph if and only if that graph contains some $H' \in \mathcal{H}$. There is some $t \leqslant |H|$ such that $H \subset H'(t)$ for every $H' \in \mathcal{H}$.

Fix $H' \in \mathcal{H}$. The graph $G$ is $H$-free so does not contain $H'(t)$. By Erdős's result on the extremal function for complete $\ell$-uniform $\ell$-partite hypergraphs [Erd64], $G$ must contain $o(n^{|H'|})$ copies of $H'$ (see for example [ABG+17, Lemma 6.2]). By the graph removal lemma (see for example [KS96, Theorem 2.9]), $G$ can be made $H'$-free by deleting $o(n^2)$ edges. As $\mathcal{H}$ is finite, there is a spanning subgraph $G'$ of $G$ with $e(G) - e(G') = o(n^2)$ which contains no $H' \in \mathcal{H}$.

Take $G'$ and sequentially delete vertices of degree less than $(c-\gamma)n$ until no more remain. Provided $n$ is large enough, so that $(e(G) - e(G'))/n^2$ is sufficiently small, this process will terminate with the deletion of at most $o(n)$ vertices. Let the resulting graph be $G''$. Then $G''$ satisfies $\delta(G'') \geqslant (c-\gamma)|G''|$ and $G''$ contains no $H' \in \mathcal{H}$. In particular, $H$ is not homomorphic to $G''$. But $c - \gamma > \delta_\chi(H\text{-hom},k)$ so $G''$ is $k$-colourable. Furthermore $G''$ was obtained from $G$ by the deletion of $o(n^2)$ edges. Thus $\delta^*_\chi(H,k) \leqslant c$, as required. □

## 5.6 Concluding Remarks

While Theorem 5.3 determines $\delta_H$ for most $H$, many minimum degree stability questions remain. Firstly how do those $\delta_H$ not determined by the theorem behave? Is $1 - 1/(r-1)$ the only accumulation point of $\{\delta_H \colon \chi(H) = r+1\}$ (as it is for $r = 2$) or does something more exotic happen? To extend Theorem 5.3 one would need to extend Theorem 4.20

below $1 - 1/(a + 8/7)$. This was the culmination of our understanding of locally bipartite and $b$-partite graphs – we would need to extend our knowledge of the structure of locally bipartite graphs, Theorem 3.2, below $8/15$ as well as the $(b+1)$-threshold of locally $b$-partite graphs, Theorem 4.4, below $1 - 1/(b + 1/7)$. All the graphs appearing in Theorem 5.3 are either an $(r-2)$-clique joined to an odd cycle or an $(r-3)$-clique joined to a 4-chromatic locally bipartite graphs. Are these the only graphs that appear? The motivation in § 5.1.2 suggests that this is the case for $r = 3$: the major obstacles to being close to tripartite are containing either a blow-up of an odd wheel of the blow-up of some 4-chromatic locally bipartite graph. However, for greater $r$ other graphs could appear. For example, containing a blow-up of some 5-chromatic locally tripartite graph would be an obstacle for being close to 4-partite. For such a graph to be relevant it would need to beat $2/3$: this reasserts the interest of Question 4.19.

In § 5.1.1 we placed $\delta_H$ as the first interesting threshold within the approximate chromatic profile. What happens next, i.e. how does $\delta^*_\chi(H, \chi(H))$ behave? For triangles (and cliques) we do have structure below $\delta_{K_3} = 2/5$ as shown in Table 3.1. Now

$$\inf_k \delta^*_\chi(H, k) \leqslant \inf_k \delta_\chi(H, k) = \delta_\chi(H),$$

and the chromatic threshold of $H$-free graphs is determined in [ABG+13]. For many $H$, $\delta_\chi(H) < \delta_H$ so $n$-vertex $H$-free graphs with minimum degree just less than $\delta_H n$ do have structure – they are close to being $k$-colourable for some bounded $k$.

Another natural direction is to consider the number of edges that need deleting to make an $n$-vertex $H$-free graph, $G$, with minimum degree at least $(\delta_H + \varepsilon)n$ $r$-partite. Are $o(n^2)$ edges really required, or, as is often the case, can one get away with $\mathcal{O}(n^{2-\rho})$ for some $\rho = \rho(H) > 0$? This has precedent. Erdős and Simonovits showed that the $H$-free graph with most edges can be made $r$-partite by deleting $\mathcal{O}(n^{2-\rho})$ edges. Also Alon and Sudakov's result, Theorem 5.1, gives an affirmative answer for $H = K_{r+1}(t)$. The heuristic here is that if more than $n^{2-\rho}$ edges are required, then, by the theorem of Kővári, Sós, and Turán, $G$ is an $r$-partite graph with some large $K_{t,t}$ appearing inside one of the parts. Joining these together ought to give some blow-up of an $F_g$ and so a copy of $H$. In both the cases of Erdős-Simonovits and Alon-Sudakov, the minimum degree of $G$ was large and so $G$ was well connected. For our present situation the following would be the most basic question.

**Question 5.13.** *For positive integers $g$ and $t$ is there some $\rho > 0$ such that every $n$-vertex graph with minimum degree at least $(2/(2g+1) + \varepsilon)n$ either contains $C_{2g-1}(t)$ or can be made bipartite by deleting $\mathcal{O}(n^{2-\rho})$ edges?*

Although this seems plausible (and indeed can be shown for $g = 3$) the smaller minimum degree condition casts doubt. Loosening the minimum degree can introduce interesting examples: there are $n$-vertex graphs with minimum degree $n/6$ that require the deletion of $n^{2-o(1)}$ edges to become bipartite but where each edge lies in at most one triangle (and so the graph is $C_3(2)$-free). Indeed, Ruzsa and Szemerédi [RS78] found, using a construction of Behrend [Beh46], a tripartite graph $G'$ (whose classes $V_1$, $V_2$, $V_3$ all have size $n$) with $n^{2-o(1)}$ edges and where every edge is in exactly one triangle. Consider taking $G'$, adding independent sets $U_1$, $U_2$, $U_3$ all of size $n$ and all edges between $U_i$ and $V_i$ for $i = 1, 2, 3$. The resulting graph $G$ has $6n$ vertices, minimum degree $n$ and every edge is in at most one triangle. Further, to make $G$ bipartite requires the deletion of all triangles from $G'$ so the deletion of at least $e(G')/3 = n^{2-o(1)}$ edges.

# Chapter 6

# Sparse Analogues

In this chapter, we make a brief foray into random graph theory, proving sparse analogues of some of the main results from Chapters 3 to 5. The ideas and proofs are most similar to the arguments of Chapter 5, hence this chapter's deferment.

## 6.1 Introduction

Classical extremal problems commonly ask about (spanning) subgraphs of the host graph $K_n$. For example, if such a subgraph is $H$-free, then what is the greatest number of edges it can have? In the last decade or so, there has been significant progress on analogues of these classical extremal results where, in place of $K_n$ as host, ones uses the binomial random graph $G(n, p)$. This is a graph on $n$ vertices where each possible edge (there are $\binom{n}{2}$) is present independently with probability $p$. These new problems are *sparse* or *random analogues* of the originals.

A sparse regularity lemma has been developed as well as a sparse counting lemma, the now-proved KŁR conjecture of Kohayakawa, Łuczak, and Rödl [KŁR97]. In this section, we use this machinery to prove random analogues of our results for minimum-degree stability and locally colourable graphs. The process of transferring results from the dense to the sparse setting has been well understood (for some excellent surveys, see Conlon [Con14], and Rödl and Schacht [RS13]). In § 6.2, we explain this process while discussing sparse analogues of minimum-degree stability from Chapter 5. Our results on locally colourable graphs from Chapters 3 and 4 require more attention: much of the current machinery considers forbidding finite families of graphs rather than infinite ones. We will pay special attention to the required adaptations and avoid labouring the more standard arguments.

In Chapter 5, we asked, given an $(r + 1)$-chromatic graph $H$, for the smallest $c$ such that any $n$-vertex $H$-free graph with minimum degree $cn$ is close to $r$-colourable. Further, in

§ 5.1.1, we generalised this to the approximate chromatic profile of $H$-free graphs

$$\delta^*_\chi(H,k) = \inf\{c\colon \text{if } |G| = n,\ \delta(G) \geqslant cn, \text{ and } G \text{ is } H\text{-free}, \\ \text{then } G \text{ can be made } k\text{-colourable by deleting } o(n^2) \text{ edges}\}.$$

The sparse analogue of this asks for the smallest $c$ such that with high probability all spanning $H$-free subgraphs of $G(n,p)$ with minimum degree $cpn$ can be made $k$-colourable by deleting $o(pn^2)$ edges. The presence of $pn$ and $pn^2$ is due to those being (the growth rate of) the expected degree and number of edges in $G(n,p)$, respectively. To this end, we make the following definition.

**Definition 6.1** (sparse approximate chromatic profile)**.** *Let $H$ be graph, $k$ a positive integer, and $p = p(n) \in [0,1]$ a functions. Define $\delta^*_\chi(H,k,p)$ to be the infimum over all $c$ for which the following holds asymptotically almost surely.*

*Every spanning $H$-free subgraph of $G(n,p)$ with minimum degree at least $cpn$ can be made $k$-colourable by deleting $o(pn^2)$ edges.*

Taking $p = 1$ recovers the approximate chromatic profile, $\delta^*_\chi(H,k)$. The chromatic threshold version of this was considered by Allen, Böttcher, Griffiths, Kohayakawa, and Morris [ABG+17, § 1.3]. They noted a threshold behaviour: for $p$ above the threshold, the sparse problem behaved just like the dense one and, for $p$ below (where $G(n,p)$ is essentially $H$-free), the problem is just a question about generic subgraphs of $G(n,p)$. Our result shows a similar threshold behaviour – to state it we need the following.

**Definition 6.2** (2-density)**.** *For a graph $H$, the **2-density of** $H$ is*

$$m_2(H) = \max\left\{\frac{e(H')-1}{|H'|-2} \colon H' \subset H, |H'| \geqslant 3\right\}.$$

The relevance of $m_2(H)$ is that $p = n^{-\frac{|H|-2}{e(H)-1}}$ satisfies $p^{e(H)}n^{|H|} = pn^2$ and so $G(n,p)$ contains a comparable number of copies of $H$ and edges. Thus if $p = cn^{-\frac{|H|-2}{e(H)-1}}$ for some small $c > 0$, then deleting a tiny proportion of the edges can remove all copies of $H$. For any subgraph $H'$ of $H$, removing all copies of $H'$ would leave an $H$-free graph and so we take a maximum. Threshold behaviour for minimum-degree stability occurs at $p = n^{-1/m_2(H)}$. For $p = \omega(n^{-1/m_2(H)})$, the machinery of Szemerédi's regularity lemma and the counting lemma still function and so the arguments of Chapter 5 push through – changing the host graph to $G(n,p)$ has had little effect. For $p = o(n^{-1/m_2(H)})$, we will show that there are $H$-free spanning subgraphs of $G(n,p)$ with minimum degree $(1-o(1))pn$ and so the question becomes how small a chromatic number can be achieved by deleting $o(pn^2)$ edges from $G(n,p)$. If $p = \omega(\log n/n)$ also, then the chromatic number will remain arbitrarily large and so there is no real stability here.

**Theorem 6.3.** *For a graph $H$, a positive integer $k$, and a function $p = p(n) \in [0,1]$,*

$$\delta^*_\chi(H,k,p) = \begin{cases} \delta^*_\chi(H,k) & \text{if } p = \omega(n^{-1/m_2(H)}), \\ 1 & \text{if } \omega(\log n/n) = p = o(n^{-1/m_2(H)}). \end{cases}$$

For $p = o(\log n/n)$ the situation degenerates: $G(n,p)$ will almost surely contain some isolated vertices, so every spanning subgraph will have minimum degree zero.

As a final remark on sparse analogues for $H$-free graphs, it is interesting to ask how many edges really need deleting in order to obtain a $k$-colourable graph: can $o(pn^2)$ in the definition of $\delta^*_\chi(H,k,p)$ be replaced by something smaller? For the special case of $H = K_3$, Allen, Böttcher, Kohayakawa, and Roberts [ABKR18] showed that $o(pn^2)$ can be replaced with $\mathcal{O}(n/p)$ and this is tight up to a constant factor.

We now move to locally colourable graphs. For a family of graphs $\mathcal{F}$ its sparse approximate chromatic profile, $\delta^*_\chi(\mathcal{F},k,p)$, is defined exactly as one would expect with the condition '$H$-free' in Definition 6.1 replaced by 'in $\mathcal{F}$'. When $p = 1$ we again recover the approximate chromatic profile, $\delta^*_\chi(\mathcal{F},k)$. For families that are closed under blow-ups (such as $a$-locally $b$-partite graphs) these are related further, continuing the theme of these families having particularly well-behaved profiles. We remind the reader that the first equality is Theorem 5.4.

**Theorem 6.4.** *Let $\mathcal{F}$ be a family of graphs, $k$ a positive integer, and $p = p(n) \in [0,1]$ a function. If $\mathcal{F}$ is closed under taking blow-ups and $p = \omega(\log n/n)$, then*

$$\delta_\chi(\mathcal{F},k) = \delta^*_\chi(\mathcal{F},k) \leqslant \delta^*_\chi(\mathcal{F},k,p).$$

Thus the approximate chromatic profile provides a lower bound for the sparse one. Theorem 6.3 shows that, for $H$-free graphs, above a threshold these are equal while degeneracy occurs below. Now locally bipartite graphs are exactly those with no odd wheels and $m_2(W_{2s+1}) = 2 + 1/(2s)$, so threshold behaviour around $n^{-1/2}$ is expected, and realised.

**Theorem 6.5.** *For a positive integer $k$, and a function $p = p(n) \in [0,1]$,*

$$\delta^*_\chi(\mathcal{F}_{1,2},k,p) = \begin{cases} \delta_\chi(\mathcal{F}_{1,2},k) & \text{if } p \geqslant n^{-1/2+o(1)}, \\ 1 & \text{if } \omega(\log n/n) = p \leqslant (2n)^{-1/2}. \end{cases}$$

This shows that for $p \leqslant (2n)^{-1/2}$ questions about locally bipartite subgraphs of $G(n,p)$ are essentially questions about $G(n,p)$. For $p \geqslant n^{-1/2+o(1)}$ the theorem together with Theorem 3.2 gives $\delta^*_\chi(\mathcal{F}_{1,2}, 3, p) = 4/7$ and $\delta^*_\chi(\mathcal{F}_{1,2}, 4, p) \leqslant 6/11$. Using our methods it is routine to obtain a result analogous to Theorem 3.2, giving structural properties of spanning locally bipartite subgraphs of $G(n,p)$ with minimum degree down to $(8/15 + o(1))pn$. We refrain from doing so.

Finally, consider locally $b$-partite graphs for $b \geqslant 3$. Theorem 6.4 gives a lower bound for their sparse profile. Also, as $\mathcal{F}_{1,2} \subset \mathcal{F}_{1,b}$, we have another lower bound $\delta^*_\chi(\mathcal{F}_{1,b}, k, p) \geqslant \delta^*_\chi(\mathcal{F}_{1,2}, k, p)$. In particular, for small enough $p$ (certainly any $p \leqslant (2n)^{-1/2}$) degeneracy occurs. For large enough $p$ there is no degeneracy: in Chapter 4 we proved that $\delta_\chi(\mathcal{F}_{1,b}, b+1) \leqslant 1 - 1/(b + 1/7)$ and we do similar here.

**Theorem 6.6.** *Let $b \geqslant 3$ be an integer and let $b' = (b+15)/2 - 90/(b+12)$. If $p = \omega(n^{-1/b'})$ and $G$ is a locally $b$-partite spanning subgraph of $G(n,p)$ with $\delta(G) \geqslant (1 - 1/(b + 1/7))pn$, then $G$ can be made $(b+1)$-colourable by deleting $o(pn^2)$ edges.*

The slightly peculiar $b'$ is an artefact of the counting lemma (Lemma 6.9 below) – it is based on the 2-densities of the graphs we need to forbid from the reduced graph.

One would expect a threshold above which the sparse profile matches the dense one and below which degeneracy occurs. Unlike for locally bipartite graphs, there is not a nice characterisation of locally $b$-partite graphs, so it is hard to even propose a good conjecture for where a threshold might be.

### 6.1.1 Tools

Just as Szemerédi's regularity lemma [Sze78] is a fundamental tool in extremal graph theory, there is an analogous sparse regularity lemma first noted independently by Kohayakawa [Koh97] and Rödl, which has become crucial for sparse extremal theory. The nomenclature is very similar.

Let $(X, Y)$ be a pair of vertex subsets of a graph $G$. We remind the reader that $d(X,Y) = e(X,Y)|X|^{-1}|Y|^{-1}$ is the density between $X$ and $Y$. The pair $(X,Y)$ is $(\varepsilon, p)$-*regular* if

$$|d(U,V) - d(X,Y)| \leqslant \varepsilon p,$$

for all $U \subset X$, $V \subset Y$ with $|U| \geqslant \varepsilon|X|$ and $|V| \geqslant \varepsilon|Y|$. A partition $\mathcal{P} = V_0 \cup V_1 \cup \cdots \cup V_k$ of $V(G)$ is an $(\varepsilon, p)$-*regular partition* if:

- $V_1, V_2, \ldots, V_k$ all have equal size and $|V_0| \leqslant \varepsilon|G|$.
- For all but at most $\varepsilon\binom{k}{2}$ pairs $ij$ ($i < j$), the pair $(V_i, V_j)$ is $(\varepsilon, p)$-regular.

The Kohayakawa-Rödl sparse regularity lemma applies to the quite general class of upper-uniform graphs [KR03]. For our purposes we only need the following special case which applies to spanning subgraphs of the random graph $G(n, p)$.

**Lemma 6.7** (sparse regularity lemma for $G(n, p)$)**.** *Let $\varepsilon > 0$ and $\ell$ be a positive integer. There is a positive integer $L$ such that the following holds asymptotically almost surely for any probability $p = p(n) = \omega(\log n/n)$. Any spanning sugraph, $G$, of $G(n, p)$ has an $(\varepsilon, p)$-regular partition into at least $\ell$ but at most $L$ parts.*

As for the dense regularity lemma, it is often convenient to discuss a reduced graph. Fix $\varepsilon > 0$, $p \in [0, 1]$ and some $\lambda \geqslant 0$ and suppose a graph $G$ has some $(\varepsilon, p)$-regular partition $\mathcal{P} = V_0 \cup V_1 \cup \cdots \cup V_k$. These induce a *reduced graph* $R(\mathcal{P}, \varepsilon, p, \lambda)$: this has vertex set $[k] = \{1, 2, \ldots, k\}$ with vertex $i$ joined to vertex $j$ exactly if the pair $(V_i, V_j)$ is $(\varepsilon, p)$-regular and $d(V_i, V_j) \geqslant \lambda p$. Note when $\lambda = 0$ this graph will have at least $(1 - \varepsilon)\binom{k}{2}$ edges by the definition of an $(\varepsilon, p)$-regular partition.

To apply all our results on locally colourable graphs we need the following minimum degree form of the sparse regularity lemma (for a proof see the appendix of [BKT10]).

**Lemma 6.8** (sparse regularity lemma, minimum degree form)**.** *Let $\varepsilon > 0$ and $\ell$ be a positive integer. There is a positive integer $L$ such that for all $\delta, \lambda, p \in [0, 1]$ the following holds asymptotically almost surely, provided $p = p(n) = \omega((\log n)^4/n)$. Any spanning subgraph, $G$, of $G(n, p)$ with*

$$\delta(G) \geqslant \delta pn$$

*has an $(\varepsilon, p)$-regular partition $\mathcal{P} = V_0 \cup V_1 \cup \cdots \cup V_k$ with $\ell \leqslant k \leqslant L$ such that the reduced graph $R = R(\mathcal{P}, \varepsilon, p, \lambda)$ satisfies*

$$\delta(R) \geqslant (\delta - \varepsilon - \lambda)|R|.$$

Note that the reduced graph is dense (its edge density depends upon $\delta$ and not $p$) and so we may apply (dense) extremal results to give it some structure. We will lift this structure to the host graph using a simplified version of the counting lemma of Conlon, Gowers, Samotij, and Schacht [CGSS14, Theorem 1.6(i)].

**Lemma 6.9** (sparse counting lemma)**.** *Let $H$ be a graph* (*on vertex set* $\{1, 2, \ldots, |H|\}$) *and $\lambda > 0$. Then there is $\varepsilon > 0$ such that for all $\eta > 0$, there is $C > 0$ such that if $p \geqslant Cn^{-1/m_2(H)}$, then asymptotically almost surely the following holds.*

*Suppose $V_1, V_2, \ldots, V_{|H|}$ are disjoint vertex sets each of size at least $\eta n$ and $G \subset G(n, p)$ is a graph on $V_1 \cup \cdots \cup V_{|H|}$ with $(V_i, V_j)$ $(\varepsilon, p)$-regular of density at least $\lambda p$ for each $ij \in E(H)$.*

*Then $G$ contains $H$.*

This theorem essentially says that if the reduced graph $R$ contains $H$, then $G$ must also contain $H$. The full result says further that the number of $H$ present is $\Omega_{\lambda,\eta}(p^{e(H)}n^{|H|})$. We will not require this strengthening.

We make use of standard facts about the random graph $G(n,p)$, most of which follow from the following standard lemma (see for example Bollobás [Bol01, § 2.3]).

**Lemma 6.10** (degrees in $G(n,p)$)**.** *For every $\eta > 0$ and $p = \omega(\log n/n)$, asymptotically almost surely the following holds in $G(n,p)$. For any vertex $v$ and all disjoint vertex sets $U$, $V$ with $U$ having size at least $\eta n$,*

$$\begin{aligned} \big||\Gamma(v) \cap U| - p|U|\big| &\leqslant \eta p|U|, \\ \big|e(U) - p\tbinom{|U|}{2}\big| &\leqslant \eta p\tbinom{|U|}{2}, \\ |d(U,V) - p| &\leqslant \eta p. \end{aligned}$$

The following construction will be useful for various lower bounds.

**Proposition 6.11.** *Let $H$ be a fixed non-$r$-colourable graph on vertex set $\{1, 2, \ldots, |H|\}$ and let $p = \omega(\log n/n)$. Partition the vertices of $G(n,p)$ into $|H|$ parts $V_1, V_2, \ldots, V_{|H|}$ as equal in size as possible. Let $G$ be the spanning subgraph of $G(n,p)$ obtained by deleting all edges within parts and all edges between $V_i, V_j$ when $ij \notin E(H)$. Then $G$ is a blow-up of $H$ and for any $\beta < \delta(H)|H|^{-1}$ asymptotically almost surely,*

$$\delta(G) > \beta p n.$$

*Furthermore, $\Omega_H(pn^2)$ edges of $G$ need deleting before it becomes $r$-colourable.*

*Proof.* Plainly $G$ is a blow-up of $H$. For any $v \in V_i$, the degree of $v$ in $G$ is equal to

$$\Big|\Gamma_{G(n,p)}(v) \cap U\Big|,$$

where $U = \cup_{ji \in E(H)} V_j$. Note that $|U| \geqslant \delta(H)|H|^{-1}n$. If $\delta(H) = 0$, then $\delta(G) \geqslant 0 > \beta p n$ and otherwise, for $\eta > 0$ sufficiently small (not depending upon $n$), Lemma 6.10 gives

$$\delta(G) \geqslant (1-\eta)p\delta(H)|H|^{-1}n > \beta p n.$$

Let $U_1 \cup \cdots \cup U_r$ be any partition of $V(G)$. It suffices to show that some $U_j$ contains

$\Omega_H(pn^2)$ edges. Say that $U_j$ *meets* $V_i$ if

$$|U_j \cap V_i| \geqslant \frac{n}{2r|H|}.$$

Now, for all $i$, $|V_i| \geqslant \lfloor n/|H| \rfloor$, so every $V_i$ is met by at least one $U_j$. As $H$ is not $r$-colourable, there must be an edge $ii'$ of $H$ such that some $U_j$ meets both $V_i$ and $V_{i'}$. Let $A = U_j \cap V_i$ and $B = U_j \cap V_{i'}$. By the definition of $G$, the edges between $A$ and $B$ in $G$ are exactly the same as those between $A$ and $B$ in $G(n,p)$. By Lemma 6.10, $d(A,B) \geqslant p/2$ and so

$$e(U_j) \geqslant e(A,B) \geqslant p/2 \cdot \left(\frac{n}{2r|H|}\right)^2 = \Omega_H(pn^2). \qquad \square$$

Finally, we will need a beautiful lemma of Erdős and Tetali [ET90] which gives exponential decay for the appearance of edge-disjoint graphs.

**Lemma 6.12** (Erdős-Tetali). *Let $X_1, X_2, \dots, X_N$ be independent indicator random variables and define for $F \subset [N]$: $X_F = \prod_{i \in F} X_i$. For $\mathcal{F} \subset \mathcal{P}([N])$ define*

$$X = \sum_{F \in \mathcal{F}} X_F = |\{F \in \mathcal{F} \colon X_F = 1\}|,$$
$$\widetilde{X} = \max\{m \colon \text{there are pairwise disjoint } F_1, \dots, F_m \in \mathcal{F} \text{ with } X_{F_i} = 1 \text{ for } 1 \leqslant i \leqslant m\}.$$

*For every non-negative integer $k$,*

$$\mathbb{P}(\widetilde{X} \geqslant k) \leqslant \frac{\mathbb{E}(X)^k}{k!}.$$

For our purposes, $N = \binom{n}{2}$, $X_i$ indicates whether or not the $i^{\text{th}}$ edge of $G(n,p)$ is present. Then $X_F$ indicates whether the subgraph of $K_n$ corresponding to the edges of $F$ is present in $G(n,p)$. In this framework, given a collection, $\mathcal{F}$, of subgraphs of $K_n$, $X$ is the number of such graphs present in $G(n,p)$ while $\widetilde{X}$ is the size of the largest pairwise edge-disjoint family of such graphs in $G(n,p)$.

## 6.2 $H$-free Graphs

Here we give the proof of Theorem 6.3 exhibiting the procedure to transfer dense results to the sparse setting. This will put us in good stead for the slightly more nuanced arguments of § 6.3. One direction is particularly clean.

**Theorem 6.13.** *Let $H$ be a graph, $k$ a positive integer, and $p = p(n) \in [0,1]$ a function. If $p = \omega(\log n/n)$, then*

$$\delta^*_\chi(H,k) \leqslant \delta^*_\chi(H,k,p).$$

*Proof.* Let $\gamma > 0$. By Theorem 5.5, $\delta^*_\chi(H,k) = \delta_\chi(H\text{-hom},k)$ and so there is a graph $H'$ with $\delta(H')|H'|^{-1} > \delta^*_\chi(H,k) - \gamma/2$ which is not $k$-colourable and to which $H$ is not homomorphic.

Use the construction of Proposition 6.11 to obtain a spanning subgraph $G$ of $G(n,p)$ which is a blow-up of $H'$, satisfies

$$\delta(G) > (\delta^*_\chi(H,k) - \gamma)pn$$

and requires the deletion of $\Omega(pn^2)$ edges to be made $k$-colourable. Since $H$ is not homomorphic to $H'$, $G$ is $H$-free. Thus

$$\delta^*_\chi(H,k,p) \geqslant \delta^*_\chi(H,k) - \gamma,$$

as required. □

We now consider the other direction for $p = \omega(n^{-1/m_2(H)})$ completing that portion of Theorem 6.3. This is a matter of using the sparse regularity lemma to obtain a dense reduced graph which is $H$-free by the sparse counting lemma. If it is also $k$-colourable, then the original spanning subgraph of $G(n,p)$ must be close to being so.

*Proof of Theorem 6.3 for $p = \omega(n^{-1/m_2(H)})$.* Let $p = \omega(n^{-1/m_2(H)})$ and $\gamma > 0$. We are left to show that $\delta^*_\chi(H,k,p) \leqslant \delta^*_\chi(H,k) + \gamma$. Let $G$ be any $H$-free spanning subgraph of $G(n,p)$ with

$$\delta(G) \geqslant (\delta^*_\chi(H,k) + \gamma)pn.$$

Fix $\lambda > 0$ small and let $\ell$ be a sufficiently large positive integer (with $\ell \geqslant 1/\lambda$) and $\varepsilon > 0$ be sufficiently small (in particular, $\varepsilon \leqslant \lambda$ and the sparse counting lemma, Lemma 6.9, applies). We will assume throughout that $n$ is sufficiently large. Applying the minimum degree version of the sparse regularity lemma, Lemma 6.8, to $G$ gives an $(\varepsilon, p)$-regular partition $\mathcal{P} = V_0 \cup V_1 \cup \cdots \cup V_k$ of $G$ with $\ell \leqslant k \leqslant L = L(\varepsilon,\ell)$ such that the reduced graph $R = R(\mathcal{P}, \varepsilon, p, \lambda)$ satisfies

$$\delta(R) \geqslant (\delta^*_\chi(H,k) + \gamma - \varepsilon - \lambda)|R| > (\delta^*_\chi(H,k) + \gamma/2)|R|.$$

First suppose that $R$ contains a copy of $H$. Each $V_i$ has size at least $n/(2L)$ and so, by the sparse counting lemma, there is some $C > 0$ such that if $p \geqslant Cn^{-1/m_2(H)}$, then $G$

contains $H$. Now $p = \omega(n^{-1/m_2(H)})$ so this does occur, contradicting the $H$-freeness of $G$. Thus $R$ is $H$-free and so, by the definition of $\delta^*_\chi(H,k)$, $R$ can be made $k$-colourable by deleting $o(|R|^2)$ edges.

By taking $\ell$ sufficiently large at the start we can assume that deleting $\lambda|R|^2$ edges from $R$ gives a $k$-colourable graph $R'$. If we delete from $G$ all edges incident to $V_0$, all edges within any $V_i$, all edges between parts $(V_i, V_j)$ which are either not $(\varepsilon, p)$-regular, have density less than $\lambda p$, or have $ij \in E(R) \setminus E(R')$, then we obtain a graph which is a blow-up of $R'$ and hence is $k$-colourable. The number of edges deleted in this way is at most

$$e(V_0, G) + \sum_{i=1}^{k} e(V_i) + \sum_{\substack{ij \text{ not} \\ (\varepsilon,p)\text{-regular}}} e(V_i, V_j) + \sum_{\substack{ij: \\ d(V_i,V_j)<\lambda p}} e(V_i, V_j) + \sum_{ij \in E(R)\setminus E(R')} e(V_i, V_j).$$

Now, as $V_i \geqslant n/(2L)$ for $i = 1, 2, \ldots, k$, Lemma 6.10 gives $e(V_i) \leqslant p|V_i|^2$ and $e(V_i, V_j) \leqslant 2p|V_i||V_j|$ for the second, third and fifth sums. It also gives $e(V_0, G) \leqslant 2p|V_0|n \leqslant 2\varepsilon pn^2$. Hence, the number of edges deleted is at most

$$\begin{aligned}
&2\varepsilon pn^2 + p\sum_{i=1}^{k}|V_i|^2 + 2p \sum_{\substack{ij \text{ not} \\ (\varepsilon,p)\text{-regular}}} |V_i||V_j| \\
&\qquad + \lambda p \sum_{\substack{ij: \\ d(V_i,V_j)<\lambda p}} |V_i||V_j| + 2p \sum_{ij \in E(R)\setminus E(R')} |V_i||V_j| \\
&\leqslant 2\varepsilon pn^2 + pk\left(\tfrac{n}{k}\right)^2 + 2p\varepsilon\tbinom{k}{2}\left(\tfrac{n}{k}\right)^2 + \lambda p\tbinom{k}{2}\left(\tfrac{n}{k}\right)^2 + 2p\lambda k^2\left(\tfrac{n}{k}\right)^2 \\
&\leqslant pn^2\left(2\varepsilon + \tfrac{1}{k} + \varepsilon + \lambda/2 + \lambda\right) \\
&\leqslant pn^2\left(5\lambda + \tfrac{1}{\ell}\right) \leqslant 6\lambda pn^2.
\end{aligned}$$

In particular, $G$ can be made $k$-colourable by deleting $o(pn^2)$ edges and so $\delta^*_\chi(H,k,p) \leqslant \delta^*_\chi(H,k) + \gamma$. □

The remaining portion of Theorem 6.3 is $\delta^*_\chi(H,k,p) = 1$ for $p$ below the threshold. This is immediate from the following (notice that Lemma 6.10 gives $\delta(G(n,p)) = (1-o(1))pn$).

**Theorem 6.14.** *Let $H$ be a graph and $p = p(n) \in [0,1]$ function with $\omega(\log n/n) = p = o(n^{-1/m_2(H)})$. Then asymptotically almost surely $G(n,p)$ has a spanning $H$-free subgraph $G$ with*

$$\delta(G(n,p)) - \delta(G) = o(pn).$$

*Furthermore, if $\varepsilon > 0$, then for all sufficiently large $n$, every graph obtained by deleting at most*

*$\varepsilon pn^2$ edges from $G$ has chromatic number greater than $1/(2\sqrt{\varepsilon})$.*

*Proof.* Let $H'$ be a subgraph of $H$ with $m_2(H) = \frac{e(H')-1}{|H'|-2}$. We define the following random variables where $v$ is a vertex of $G(n,p)$.

- $X$ is the number of copies of $H'$ in $G(n,p)$.
- $X(v)$ is the number of copies of $H'$ in $G(n,p)$ that contain $v$.
- $\widetilde{X}(v)$ is the size of the largest collection of edge-disjoint copies of $H'$ in $G(n,p)$ that all contain $v$.

First note that

$$\mathbb{E}(X) \leqslant n^{|H'|} p^{e(H')} = o(pn^2).$$

Next, $\sum_v X(v) = |H'|X$ and so, by symmetry,

$$\mathbb{E}(X(v)) = \left|H'\right| \cdot \mathbb{E}(X)/n = o(pn).$$

Now, using the inequality $k! \geqslant (k/e)^k$ and Lemma 6.12, for any integer $k$

$$\mathbb{P}\big(\widetilde{X}(v) \geqslant k\big) \leqslant \left(\frac{e\mathbb{E}(X(v))}{k}\right)^k.$$

Thus, if we take $k \geqslant 6 \cdot \mathbb{E}(X(v))$, then $\mathbb{P}(\widetilde{X}(v) \geqslant k) \leqslant 2^{-k}$. Take $k$ to be the larger of $2\log n$ and $6 \cdot \mathbb{E}(X(v))$. A union bound gives

$$\mathbb{P}\big(\text{there is some } v : \widetilde{X}(v) \geqslant k\big) \leqslant n \cdot 2^{-k} = o(1),$$

and so asymptotically almost surely $G(n,p)$ has $\widetilde{X}(v) \leqslant k$ for all $v$. Note further that $k = o(pn)$, since $\mathbb{E}(X(v)) = o(pn)$ and $p = \omega(\log n/n)$.

Let $H_1, H_2, \ldots, H_m$ be a maximal collection of edge-disjoint copies of $H'$ in $G(n,p)$. Let $G$ be the graph obtained from $G(n,p)$ by deleting all the edges of $H_1, H_2, \ldots, H_m$. By maximality, $G$ contains no copies of $H'$ and so is $H$-free. Consider a vertex $v$. The number of $H_i$ that contain $v$ is at most $\widetilde{X}(v) \leqslant k$. In particular, the number of deleted edges incident to $v$ is at most $e(H')k = o(pn)$ and so

$$\delta(G(n,p)) - \delta(G) = o(pn).$$

Now let $G'$ be a graph obtained from $G$ by deleting at most $\varepsilon pn^2$ edges. Then $G'$ was obtained from $G(n,p)$ by deleting less than $n \cdot o(pn) + \varepsilon pn^2 \leqslant 3/2 \cdot \varepsilon pn^2$ edges provided $n$ is large enough. Let $U$ be any subset of at least $2\sqrt{\varepsilon}n$ vertices in $G(n,p)$. By

Lemma 6.10, for all large $n$,

$$e_{G(n,p)}(U) \geqslant p\binom{|U|}{2} \cdot 99/100 > p \cdot |U|^2 \cdot 49/50 > 3/2 \cdot \varepsilon p n^2,$$

and so $U$ cannot be an independent set in $G'$. Thus, $\alpha(G') < 2\sqrt{\varepsilon}n$ and so $\chi(G') > 1/(2\sqrt{\varepsilon})$. □

## 6.3 Locally Colourable Graphs

We now give the proofs of Theorems 6.4 to 6.6, highlighting where they differ from the corresponding proofs of § 6.2.

*Proof of Theorem 6.4.* Let $\gamma > 0$. By definition, there is a graph $H \in \mathcal{F}$ with $\delta(H)|H|^{-1} > \delta_\chi(\mathcal{F}, k) - \gamma/2$ which is not $k$-colourable. Use the construction of Proposition 6.11 to obtain a spanning subgraph $G$ of $G(n, p)$ which is a blow-up of $H$, satisfies $\delta(G) > (\delta_\chi(\mathcal{F}, k) - \gamma)pn$ and requires the deletion of $\Omega(pn^2)$ edges to be made $k$-colourable. Since $\mathcal{F}$ is closed under blow-ups, $G \in \mathcal{F}$. Thus

$$\delta^*_\chi(\mathcal{F}, k, p) \geqslant \delta_\chi(\mathcal{F}, k) - \gamma.$$ □

We now complete the $p \geqslant n^{-1/2+o(1)}$ portion of Theorem 6.5. This is very similar to the corresponding part of the proof of Theorem 6.3, although some care has to be taken when showing that the reduced graph cannot contain an odd wheel. Odd wheels with short rims have 2-density away from two, so Lemma 6.9 does not immediately transfer them from the reduced graph to the original. This is sidestepped by aiming for a longer odd wheel.

*Proof of Theorem 6.5 for $p \geqslant n^{-1/2+o(1)}$.* Let $\gamma > 0$ and $p \geqslant n^{-1/2+\gamma}$. It suffices to show that $\delta^*_\chi(\mathcal{F}_{1,2}, k, p) \leqslant \delta_\chi(\mathcal{F}_{1,2}, k) + \gamma$. Let $G$ be any locally bipartite spanning subgraph of $G(n, p)$ with

$$\delta(G) \geqslant (\delta_\chi(\mathcal{F}_{1,2}, k) + \gamma)pn.$$

Note that $m_2(W_{2s+1}) = 2 + 1/(2s)$. We first choose some positive integer $s$ such that both $-1/2 + \gamma > -1/m_2(W_{2s+1})$ and $2 - 1/(1/2 + \gamma/2) > 2/(2s + 1)$ hold ($s = \lceil 1/\gamma \rceil$ suffices). Set $\lambda = \gamma/4$ and let $\ell$ be a sufficiently large positive integer and $\varepsilon > 0$ be sufficiently small.

Applying the minimum degree version of the sparse regularity lemma, Lemma 6.8, to $G$ gives an $(\varepsilon, p)$-regular partition $\mathcal{P} = V_0 \cup V_1 \cup \cdots \cup V_k$ of $G$ with $\ell \leqslant k \leqslant L = L(\varepsilon, \ell)$

such that the reduced graph $R = R(\mathcal{P}, \varepsilon, p, \lambda)$ satisfies

$$\delta(R) \geqslant (\delta_\chi(\mathcal{F}_{1,2}, k) + \gamma - \varepsilon - \lambda)|R| \geqslant (\delta_\chi(\mathcal{F}_{1,2}, k) + \gamma/2)|R|.$$

By the definition of $\delta_\chi(\mathcal{F}_{1,2}, k)$, $R$ is either $k$-colourable or is not locally bipartite. We first assume that $R$ is not locally bipartite, so has some vertex $v$ such that $R_v$ contains an odd cycle. Now $\delta_\chi(\mathcal{F}_{1,2}, k) \geqslant \delta_\chi(\mathcal{F}_{1,2}) = 1/2$ so

$$\begin{aligned} \frac{\delta(R_v)}{|R_v|} &\geqslant \frac{\delta(R) - (|R| - |R_v|)}{|R_v|} \geqslant 2 - \frac{|R|}{\delta(R)} \\ &\geqslant 2 - \frac{1}{1/2 + \gamma/2} > \frac{2}{2s+1}. \end{aligned}$$

In particular, by Lemma 5.9, $R_v$ contains an odd cycle $S$ of length at most $2s+1$ and so $R$ contains an odd wheel with centre $v$ and rim $S$. We may assume the parts of $\mathcal{P}$ corresponding to $S$ are $V_1, \ldots, V_{|S|}$ in that order. Now, as $C_{2s+1}$ is homomorphic to $S$, we may split $V_1, V_2, \ldots, V_{|S|}$ to obtain vertex sets $W_1, W_2, \ldots, W_{2s+1}$ satisfying the following properties.

- For each $i = 1, 2, \ldots, 2s+1$, $W_i$ is a subset of $V_{f(i)}$ for some $f(i) = 1, 2, \ldots, |S|$.
- $|W_i| \geqslant |V_{f(i)}|/(2s+1)$.
- For each $i$, $f(i+1) = f(i) \pm 1$ where $i$ is considered modulo $|S|$ and $f(i)$ is considered modulo $2s+1$.

The point of the third condition is as follows. By the definition of $R$, for each $i$ the pair $(V_{f(i)}, V_{f(i+1)})$ is $(\varepsilon, p)$-regular of density at least $\lambda p$. But $W_i \subset V_{f(i)}$ and $W_{i+1} \subset V_{f(i+1)}$ have sizes at least $|V_{f(i)}|/(2s+1) = |V_{f(i+1)}|/(2s+1)$ and so, by $(\varepsilon, p)$-regularity, the pair $(W_i, W_{i+1})$ is $((2s+1)\varepsilon, p)$-regular with density at least $(\lambda - \varepsilon)p \geqslant \lambda p/2$. Note this holds for $i = 1, 2, \ldots, 2s+1$ (indices considered modulo $2s+1$). Let the part of $\mathcal{P}$ corresponding to $v$ be $W$. Then, by the same reasoning, the pairs $(W, W_i)$ are also $((2s+1)\varepsilon, p)$-regular with density at least $\lambda p/2$.

We now apply the sparse counting lemma, Lemma 6.9, with $H = W_{2s+1}$ and density $\lambda/2$ to the vertex sets $W$, $W_1$, $W_2$, $\ldots$, $W_{2s+1}$. Each of these vertex sets has size at least $|V_1|/(2s+1) \geqslant n/(2(2s+1)L)$ and so there is a $C$ not depending upon $n$, such that if $p \geqslant Cn^{-1/m_2(W_{2s+1})}$, then $G$ asymptotically almost surely contains $W_{2s+1}$. But $-1/m_2(W_{2s+1}) < -1/2 + \gamma$, so $p = \omega(n^{-1/m_2(W_{2s+1})})$ and so $G$ does indeed contain $W_{2s+1}$, which contradicts local bipartiteness. Hence $R$ must be locally bipartite and so is $k$-colourable.

If we delete from $G$ all edges incident to $V_0$, all edges within any $V_i$, and all edges between pairs of parts $(V_i, V_j)$ which are either not $(\varepsilon, p)$-regular or have density less than $\lambda p$, then we obtain a graph which is a blow-up of $R$ and hence is $k$-colourable. By an identical argument to the corresponding one in Theorem 6.3, the number of edges deleted in this way is at most $\gamma pn^2$. $\square$

To complete the proof of Theorem 6.5 we just need to address the behaviour for $p \leqslant (2n)^{-1/2}$. The proof is similar to that of Theorem 6.14 although we need to be careful about odd wheels of very long length.

**Theorem 6.15.** *If $p \leqslant (2n)^{-1/2}$, then asymptotically almost surely $G(n,p)$ has a locally bipartite spanning subgraph $G$ with*

$$\delta(G(n,p)) - \delta(G) < pn^{4/5} = o(pn).$$

*Furthermore, if $p = \omega(\log n/n)$ and $\varepsilon > 0$, then, for all sufficiently large $n$, every graph obtained by deleting at most $\varepsilon pn^2$ edges from $G$ has chromatic number greater than $1/(2\sqrt{\varepsilon})$.*

*Proof.* It is helpful to rewrite the upper bound on $p$ as $p^2 n \leqslant 1/2$. We define the following random variables, where $k$ is a positive integer and $v$ is a vertex of $G(n,p)$.

- $X$ is the number of odd wheels in $G(n,p)$.
- $X_k$ is the number of $(2k+1)$-wheels in $G(n,p)$.
- $X(v)$ is the number of odd wheels in $G(n,p)$ which contain $v$.
- $\widetilde{X}(v)$ is the size of the largest collection of edge-disjoint odd wheel in $G(n,p)$ which contain $v$.
- $X_k(v)$ is the number of $(2k+1)$-wheels in $G(n,p)$ which contain $v$.

Note that the total number of $(2k+1)$-wheels in an $n$-vertex complete graph is

$$\tfrac{1}{2(2k+1)} \cdot n(n-1)\cdots(n-2k-1),$$

as the automorphism group of $W_{2k+1}$ is the symmetry group of the regular $(2k+1)$-gon (a dihedral group) which has size $2(2k+1)$. In particular,

$$\mu_k := \mathbb{E}(X_k) = \tfrac{1}{2(2k+1)} \cdot n(n-1)\cdots(n-2k-1)p^{2(2k+1)} \leqslant (p^2 n)^{2k+1} n. \tag{6.1}$$

Next, $\sum_v X_k(v) = (2k+2)X_k$ and so, by symmetry and the equality in (6.1),

$$\mu_k(v) := \mathbb{E}(X_k(v)) = (2k+2)\mathbb{E}(X_k)/n \leqslant (p^2 n)^{2k+1}.$$

First suppose that $p \leqslant n^{-3/4}$. Then

$$\mathbb{E}(X) = \sum_{k\geqslant 1} \mu_k \leqslant n \cdot \sum_{k \geqslant 1} (p^2 n)^{2k+1} = n \cdot \tfrac{(p^2 n)^3}{1-(p^2 n)^2} \leqslant \tfrac{n(p^2 n)^3}{1-(1/2)^2} \leqslant 4/3 \cdot n^{-1/2} = o(1).$$

Thus, by Markov's inequality, for $p \leqslant n^{-3/4}$, $G(n,p)$ asymptotically almost surely does not contain any odd wheels and so is locally bipartite. In particular, we may take $G$ as being all of $G(n,p)$.

Now suppose that $p \geqslant n^{-3/4}$. We first show that $G(n,p)$ does not contain any $W_{2k+1}$ with $k \geqslant \log n$. Indeed the expected number of such odd wheels is

$$\begin{aligned}\sum_{k \geqslant \log n} \mu_k \leqslant n \cdot \sum_{k \geqslant \log n} (p^2 n)^{2k+1} &\leqslant n \cdot \tfrac{(p^2 n)^{2\log n+1}}{1-(p^2n)^2} \\ &\leqslant 4/3 \cdot n \cdot (1/2)^{2\log n+1} = 2/3 \cdot n^{1-2\log 2} = o(1),\end{aligned}$$

and so, by Markov's inequality, with high probability all $W_{2k+1}$ in $G(n,p)$ have $k < \log n$. We will now show that asymptotically almost surely $\widetilde{X}(v) < pn^{4/5}/(4\log n + 2)$ for every vertex $v$. Note that

$$\mu(v) := \mathbb{E}(X(v)) = \sum_{k \geqslant 1} \mu_k(v) \leqslant \sum_{k \geqslant 1} (p^2 n)^{2k+1} = \tfrac{(p^2n)^3}{1-(p^2n)^2} \leqslant \tfrac{(1/2)^3}{1-(1/2)^2} = 1/6.$$

Applying Lemma 6.12 crudely (omitting the factorial in the denominator) we have

$$\mathbb{P}\big(\widetilde{X}(v) \geqslant pn^{4/5}/(4\log n + 2)\big) \leqslant \mu(v)^{pn^{4/5}/(4\log n+2)} \leqslant 6^{-pn^{4/5}/(4\log n+2)}.$$

Taking a union bound over all vertices and using $p \geqslant n^{-3/4}$ gives

$$\begin{aligned}\mathbb{P}\big(\text{there is some } v : \widetilde{X}(v) \geqslant pn^{4/5}/(4\log n + 2)\big) &\leqslant n \cdot 6^{-pn^{4/5}/(4\log n+2)} \\ &\leqslant n \cdot 6^{-n^{1/20}/(4\log n+2)} = o(1).\end{aligned}$$

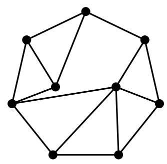
To summarise, we have shown that for $p \geqslant n^{-3/4}$ (in fact for all $p$), asymptotically almost surely $G(n,p)$ contains no $W_{2k+1}$ with $k \geqslant \log n$ and $\widetilde{X}(v) < pn^{4/5}/(4\log n + 2)$ for all $v$. Let $F_1, F_2, \ldots, F_m$ be a maximal collection of edge-disjoint odd wheels in $G(n,p)$. Let $G$ be the graph obtained from $G(n,p)$ by deleting all the edges of $F_1, F_2, \ldots, F_m$. By maximality, $G$ contains no odd wheels and so is locally bipartite. Consider a vertex $v$. The number of $F_i$ that contain $v$ is at most $\widetilde{X}(v) < pn^{4/5}/(4\log n + 2)$. As $G(n,p)$ contains no $W_{2k+1}$ with $k \geqslant \log n$, each $F_i$ has less than $4\log n + 2$ edges. In particular, the number of deleted edges incident to $v$ is less than $pn^{4/5}$ and so

$$\delta(G(n,p)) - \delta(G) < pn^{4/5} = o(pn).$$

The fact that, for $p = \omega(\log n/n)$, deleting $\varepsilon pn^2$ edges from $G$ leaves a graph with chromatic number greater than $1/(2\sqrt{\varepsilon})$ was shown in the proof of Theorem 6.14. □

All that remains is to prove Theorem 6.6. A similar argument could be used to extend this result to $a$-locally $b$-partite graphs in the vein of Theorem 4.4.

*Proof of Theorem 6.6.* Before commencing we need to define, $T$, the following 4-chromatic subgraph of $T_0$.



$T$

We remind the reader that $b' = (b+15)/2 - 90/(b+12)$ and $p = \Omega(n^{-1/b'})$. Fix $\gamma > 0$ small. It suffices to prove that any spanning subgraph of $G(n,p)$ with minimum degree at least $(1 - 1/(b+1/7) + \gamma)pn$ can be made $(b+1)$-colourable by deleting at most $\gamma p n^2$ edges. Set $\lambda = \gamma/4$ and let $\ell$ be a sufficiently large positive integer and $\varepsilon > 0$ be sufficiently small. Let $G$ be a spanning locally $b$-partite subgraph of $G(n,p)$ with

$$\delta(G) \geqslant \left(1 - \frac{1}{b+1/7} + \gamma\right) pn.$$

Applying the minimum degree version of the sparse regularity lemma, Lemma 6.8, to $G$ gives an $(\varepsilon, p)$-regular partition $\mathcal{P} = V_0 \cup V_1 \cup \cdots \cup V_k$ of $G$ with $\ell \leqslant k \leqslant L = L(\varepsilon, \ell)$ such that the reduced graph $R = R(\mathcal{P}, \varepsilon, p, \lambda)$ satisfies

$$\delta(R) \geqslant (1 - 1/(b+1/7) + \gamma - \varepsilon - \lambda) \cdot |R| > (1 - 1/(b+1/7)) \cdot |R|.$$

Suppose first that $R$ is not locally $b$-partite. We show that there is a $(b-2)$-clique $K$ in $R$ with $R_K$ not 3-colourable. Indeed, for $b = 3$, $R$ is not locally tripartite so there is a vertex in $R$ whose neighbourhood is not 3-colourable. Take $K$ to be this vertex. For $b > 3$, $R$ is not locally $b$-partite so contains a vertex $w_1$ with $R_{w_1}$ not $b$-colourable. Applying the lifting lemma, Lemma 4.6, to $\{w_1\}$ gives

$$\delta(R_{w_1}) > \left(1 - \frac{1}{b-1+1/7}\right) \cdot |R_{w_1}|.$$

By Theorem 4.4, if $R_{w_1}$ was locally $(b-1)$-partite, then it would be $b$-colourable. Hence, $R_{w_1}$ is not locally $(b-1)$-partite and so there is a vertex $w_2$ in $R_{w_1}$ with $R_{w_1,w_2}$ not $(b-1)$-colourable. Repeating this argument does indeed give a $(b-2)$-clique $K$ with $R_K$ not 3-colourable. Now, applying Lemma 4.6 to $K$ gives

$$\delta(R_K) > \left(1 - \frac{1}{b-(b-2)+1/7}\right) \cdot |R_K| = 8/15 \cdot |R_K|.$$

By Theorem 3.2, $R_K$ contains either an odd wheel, a copy of $H_2$, or a copy of $T_0$. In particular, $R_K$ contains either an odd wheel, a copy of $H_0$, or a copy of $T$.

We now bound above the order of this odd wheel (if it exists). Suppose $R_K$ contains an odd wheel, so there is some vertex $v$ such that $R_{K,v}$ contains an odd cycle. Now, Lemma 4.6 gives $\delta(R_{K,v}) > 1/8 \cdot |R_{K,v}|$ and so, by Lemma 5.9, $R_{K,v}$ contains some odd cycle of length at most 15.

Thus $R$ contains one of the following nine graphs: $K_{b-1} + C_3, K_{b-1} + C_5, \dots, K_{b-1} + C_{15}$, $K_{b-2} + H_0$, $K_{b-2} + T$. Now,

$$\begin{aligned} m_2(K_{b-2} + H_0) &= \frac{b^2 + 9b - 2}{2(b+3)} = \frac{b+6}{2} - \frac{10}{b+3} < b', \\ m_2(K_{b-2} + T) &= \frac{b^2 + 13b - 2}{2(b+5)} = \frac{b+8}{2} - \frac{21}{b+5} < b', \\ m_2(K_{b-1} + C_{2s+1}) &= \frac{b^2 + (4s-1)b}{2(b+2s-2)} = \frac{b+2s+1}{2} - \frac{(s-1)(2s+1)}{b+2s-2} \leqslant b', \end{aligned}$$

provided $s \leqslant 7$. As $p = \omega(n^{-1/b'})$, Lemma 6.9 implies that $G$ must contain one of $K_{b-1} + C_3$, $K_{b-1} + C_5$, $\dots$, $K_{b-1} + C_{15}$, $K_{b-2} + H_0$, $K_{b-2} + T$ and so $G$ is not locally $b$-partite, a contradiction.

Hence, $R$ is locally $b$-partite. Now, $\delta(R) > (1 - 1/(b + 1/7)) \cdot |R|$ and so, by Theorem 4.4, $R$ is $(b+1)$-colourable. If we delete from $G$ all edges incident to $V_0$, all edges within any $V_i$, and all edges between pairs of parts $(V_i, V_j)$ which are either not $(\varepsilon, p)$-regular or have density less than $\lambda p$, then we obtain a graph which is a blow-up of $R$ and hence is $(b+1)$-colourable. Just as in the proof of Theorem 6.3, the number of deleted edges is less than $\gamma p n^2$. □

CHAPTER 7

# THE $\chi$-RAMSEY PROBLEM FOR TRIANGLE-FREE GRAPHS

The work in this chapter was joint with Ewan Davies, initially at the March 2021 workshop 'Entropy Compression and Related Methods' organised by Ross Kang and Jean-Sébastien Sereni. Many of the results of this chapter have been submitted in a forthcoming paper [DI21].

## 7.1 INTRODUCTION

The classical Ramsey question for triangle-free graphs asks for the value of $R(3,t)$ or, equivalently, the smallest independence number amongst $n$-vertex triangle-free graphs.

The colour classes of vertex-colourings are independent sets, so the chromatic number, $\chi(G)$, and independence number, $\alpha(G)$, of a graph $G$ satisfy

$$\alpha(G) \cdot \chi(G) \geqslant |G|. \tag{7.1}$$

This suggests a natural 'chromatic Ramsey' question that Erdős [Erd67b] first asked in 1967: what is the greatest chromatic number amongst $n$-vertex triangle-free graphs? He raised the problem of determining the following two functions

$$\begin{aligned} f(n) &= \max\{\chi(G) \colon G \text{ is triangle-free, } |G| = n\}, \\ g(m) &= \max\{\chi(G) \colon G \text{ is triangle-free, } e(G) = m\}, \end{aligned}$$

and showed that and showed that

$$\begin{aligned} f(n) &= \Omega(n^{1/2}/\log n), \\ f(n) &= \mathcal{O}(n^{1/2}), \\ g(m) &= \Omega(m^{1/3}/\log m), \\ g(m) &= \mathcal{O}(m^{1/3}). \end{aligned}$$

For now we focus on $f(n)$. An early indication of its growth rate was given by Ajtai, Komlós, and Szemerédi's [AKS80] bound $R(3,t) = \mathcal{O}(t^2/\log t)$, which shows that every $n$-vertex triangle-free graph $G$ has independence number

$$\alpha(G) = \Omega(\sqrt{n \log n}),$$

and so *Hall ratio*

$$\rho(G) := \max_{\varnothing \neq H \subset G} \frac{|H|}{\alpha(H)} = \mathcal{O}(\sqrt{n/\log n}).$$

The Hall ratio is a graph parameter which, in light of inequality (7.1), satisfies $\rho(G) \leqslant \chi(G)$. For random-like graphs (which are often excellent candidate graphs for Ramsey bounds), inequality (7.1) is often not far from an equality, so the Hall ratio gives a reasonable approximation for the chromatic number. For our purposes we use the Hall ratio as a good way to compare bounds on the independence and chromatic numbers of a graph.

Erdős and Hajnal [EH85] (see [JT94, pages 124–125] for details) noted that iteratively pulling out the large independent sets guaranteed by Ajtai, Komlós, and Szemerédi's result and giving each one a different colour shows that $f(n) = \mathcal{O}(\sqrt{n/\log n})$. The matching lower bound follows from a pivotal result of Kim [Kim95] that determined the asymptotic growth of $R(3,t)$. He constructed $n$-vertex triangle-free graphs with independence number $\mathcal{O}(\sqrt{n \log n})$ and noted that

$$(1/9 - o(1))\sqrt{n/\log n} \leqslant f(n) \leqslant (2\sqrt{2} + o(1))\sqrt{n/\log n},$$

so $f(n) = \Theta(\sqrt{n/\log n})$. A recent groundbreaking result proved independently by Bohman and Keevash [BK21] and Fiz Pontiveros, Griffiths, and Morris [FGM20] gives as a corollary an improvement to the lower bound. They followed the triangle-free process to its asymptotic end, showing that there are $n$-vertex triangle graphs with Hall ratio (and so chromatic number) at least

$$(1/\sqrt{2} - o(1))\sqrt{n/\log n},$$

and so there is a factor of four between the upper and lower bounds for $f(n)$. Our first result is an improvement to the upper bound of $f(n)$ by a factor of $\sqrt{2}$.

**Theorem 7.1.** *Let $G$ be a triangle-free graph on $n$ vertices. Then*

$$\chi(G) \leqslant (2 + o(1))\sqrt{n/\log n}.$$

Cames van Batenburg, de Joannis de Verclos, Kang, and Pirot [CdKP20] recently highlighted the problem of tightening the asymptotic constants as well as considering bounds for the fractional and list chromatic numbers (these are defined in § 7.2). Table 7.1 gives a summary of the best upper bounds known in terms of $n$ for the Hall ratio, $\rho$, fractional chromatic number, $\chi_f$, chromatic number, $\chi$, and list chromatic number, $\chi_\ell$, of an $n$-vertex triangle-free graph. We remark that all graphs have $\rho \leqslant \chi_f \leqslant \chi \leqslant \chi_\ell$, a fact we prove in § 7.2.

The upper bound for $\rho$ comes from Shearer's [She83] bound $R(3,t) \leqslant (1 + o(1))t^2/\log t$ which improves the constant in Ajtai, Komlós, and Szemerédi's [AKS80] aforementioned result. Any improvement to the upper bound for $\rho$ would immediately strengthen this upper bound for $R(3,t)$. In particular, given the difficulty of improving the Ramsey number bound, we compare the upper bounds for $\chi_f$, $\chi$, $\chi_\ell$ in Table 7.1 to the bound $\sqrt{2n/\log n}$ for $\rho$. This is why the upper bounds are stated with this factor separated – we are a factor of $\sqrt{2}$ away from matching the chromatic number with the Hall ratio.

| Parameter | Previous bound | This chapter |
|---|---|---|
| $\rho(G)$ | $(1 + o(1))\sqrt{2n/\log n}$ | |
| $\chi_f(G)$ | $(\sqrt{2} + o(1))\sqrt{2n/\log n}$ | |
| $\chi(G)$ | $(2 + o(1))\sqrt{2n/\log n}$ | $(\sqrt{2} + o(1))\sqrt{2n/\log n}$ |
| $\chi_\ell(G)$ | $(1 + o(1))\sqrt{8n}$ | $(4 + o(1))\sqrt{2n/\log n}$ |

Table 7.1: Upper bounds for an $n$-vertex triangle-free graph $G$

The upper bound for the chromatic number given by Kim [Kim95] follows by applying Erdős and Hajnal's [EH85] method of iteratively pulling out the large independent sets guaranteed by Shearer's result. The bounds for the fractional and list chromatic numbers are due to Cames van Batenburg et al. [CdKP20].

In light of Theorem 7.1 there is now a gap of $\sqrt{2}$ between the upper bounds for (fractional) chromatic number and for the Hall ratio. Cames van Batenburg et al. conjectured that it is possible to remove this $\sqrt{2}$ – in § 7.5 we discuss how this might be done.

A particularly noticeable feature of the previous bounds in Table 7.1 is the upper bound for the list chromatic number, which does not have the same growth rate as the lower bound provided by the triangle-free process. We give a short argument rectifying this.

**Theorem 7.2.** *Let $G$ be a triangle-free graph on $n$ vertices. Then*

$$\chi_\ell(G) \leqslant (4\sqrt{2}+o(1))\sqrt{n/\log n}.$$

We briefly discuss $g(m)$. Kim's [Kim95] construction gives $g(m) = \Omega(m^{1/3}/(\log m)^{2/3})$ and the matching upper bound was first provided by Poljak and Tuza [PT94] (see also [GT00; Nil00]), so

$$g(m) = \Theta\big(m^{1/3}/(\log m)^{2/3}\big).$$

We give the edge question less attention, as the bounds use the vertex results. In particular, the improved upper bounds for $g(m)$ as well as for the list chromatic number of a graph with $m$ edges use Theorems 7.1 and 7.2 respectively in their proofs. The second of these upper bounds is of particular interest as it establishes the correct growth rate (the previous upper bound was just $\mathcal{O}(m^{1/3})$) confirming a conjecture of Cames van Batenburg et al. [CdKP20, Conjecture 6.1].

**Theorem 7.3.** *Let $G$ be a triangle-free graph with $m$ edges. Then*

$$\chi(G) \leqslant (3^{5/3}+o(1)) \cdot m^{1/3}/(\log m)^{2/3},$$
$$\chi_\ell(G) \leqslant (8 \cdot 6^{2/3}+o(1)) \cdot m^{1/3}/(\log m)^{2/3}.$$

For comparison, the best lower bound for $g(m)$ is

$$g(m) \geqslant \big((3/2)^{2/3}-o(1)\big)m^{1/3}/(\log m)^{2/3}$$

as given by the triangle-free process [BK21; FGM20].

## 7.2 Notation

We need to fix some notation with reference to fractional and list colourings. Given a graph $G$, a *list assignment* is a function $L\colon V(G) \to \mathcal{P}(\mathbb{Z}^+)$ – to each vertex $v$ a list of allowed colours $L(v)$ has been assigned. An *$L$-colouring of $G$* is a colouring $c\colon V(G) \to \mathbb{Z}^+$ with the following two properties.

- It is *proper*: for every edge $uv$, $c(u) \neq c(v)$.
- It is *consistent with $L$*: for every vertex $v$, $c(v) \in L(v)$.

Note that the chromatic number, $\chi(G)$, of $G$ is the least $k$ such that $G$ is $L$-colourable where $L(v) = [k]$ for every vertex $v$. The *list chromatic number*, $\chi_\ell(G)$, of $G$ is the least $k$ such that $G$ is $L$-colourable for any list assignment $L$ where every vertex $v$ has $|L(v)| \geqslant k$. The inequality $\chi \leqslant \chi_\ell$ immediately follows. The list chromatic number can be much greater than the usual one. It is folklore that the bipartite graph $K_{n,n}$ has list chromatic number $\Omega(\log n)$ (for example, see Alon [Alo92]) with the asymptotically tight $(1+o(1))\log_2 n$ shown more recently by Saxton and Thomason [ST15].

A *fractional colouring of* $G$ is a colouring $c\colon V(G) \to \mathcal{P}(\mathbb{R})$ with the following two properties.

- It is *proper*: for every edge $uv$, $c(u) \cap c(v) = \varnothing$.
- For every vertex $v$, $c(v)$ is (Lebesgue) measurable with measure at least one.

The *weight of* $c$ is the measure of $\cup_v c(v)$. The *fractional chromatic number*, $\chi_f(G)$, of $G$ is the smallest weight of any fractional colouring of $G$. If we insist that every $c(v)$ is of the form $[k, k+1)$ for some integer $k$, then we recover a standard vertex colouring. It follows that $\chi_f \leqslant \chi$. Using $\mu$ to denote the Lebesgue measure, we have

$$|G| \leqslant \sum_v \mu(c(v)) \leqslant \alpha(G) \cdot \mu\big(\cup_v c(v)\big),$$

where the last inequality is due to the fact that every real number is in at most $\alpha(G)$ of the $c(v)$. This gives $\chi_f \geqslant \rho$.

Consider picking a uniformly random real $x \in \cup_v c(v)$ and setting $\boldsymbol{I}_x = \{v\colon x \in c(v)\}$. Then $\boldsymbol{I}_x$ is a random independent set of $G$ and the probability that any vertex lies in $\boldsymbol{I}_x$ is at least $\mu(\cup_v c(v))^{-1}$. This gives an equivalent notion of fractional colouring: a fractional colouring of weight at most $k$ is a probability distribution on the independent sets of $G$ such that the random independent set $\boldsymbol{I}$ obtained and every vertex $v$ satisfy

$$\mathbb{P}(v \in \boldsymbol{I}) \geqslant 1/k.$$

## 7.3 Tools and Proof Ideas

Let $G$ be an $n$-vertex triangle-free graph and let $d$ be its average degree. The original upper bound for $R(3,t)$ (and so for the Hall ratio of $G$) of Ajtai, Komlós, and Szemerédi [AKS80] was obtained by comparing two lower bounds for the independence number of $G$. The first of these is a trivial lower bound: neighbourhoods in triangle-free graphs are independent, so $\alpha(G) \geqslant \Delta(G) \geqslant d$. The second was the real breakthrough: they showed that $\alpha(G) = \Omega(n \log d/d)$. The trivial lower bound increases with $d$ while the latter one decreases, so the worst case is when they are equal. Shearer [She83] analysed a greedy random algorithm improving the second lower bound to $\alpha(G) \geqslant (1-o(1)) \cdot n \log d/d$. This meets the trivial lower bound for $d = (1/\sqrt{2} + o(1))\sqrt{n \log n}$ thus giving the bound for $\rho(G)$ in Table 7.1 or equivalently $R(3,t) \leqslant (1+o(1)) \cdot t^2/\log t$.

In the other direction, the triangle-free process [BK21; FGM20] gives $n$-vertex triangle-free graphs in which every vertex has degree $(1/\sqrt{2}+o(1))\sqrt{n\log n}$ and the independence number is at most $(\sqrt{2}+o(1))\sqrt{n\log n}$ (which is believed to be tight). This shows $R(3,t) \geqslant (1/4-o(1)) \cdot t^2/\log t$ and gives a lower bound of $(1/\sqrt{2}-o(1))\sqrt{n/\log n}$ for the Hall ratio. The gap of four for the Ramsey number and two for the Hall ratio are equivalent – any tightening of one gives an improvement to the other. It is interesting that the gap of four for the Ramsey number seems to stem from two sources: the graph at the end of the triangle-free process is believed to have independence number which is both $(2-o(1)) \cdot n\log d/d$ (i.e. twice Shearer's bound) and roughly twice the maximum degree. Thus to bridge the gap would require both improving Shearer's bound by a factor of two (which Fiz Pontiveros, Griffiths, and Morris [FGM20] believe to be true in general) but also proving that for the extremal graphs the independence number is twice the maximum degree – a fact that is not true for general triangle-free graphs (consider, for example, the Vega graphs of § 3.1.2).

To bound $f(n)$, the method of Erdős and Hajnal [EH85] iteratively removes the large independent sets of size $(1/\sqrt{2}-o(1))\sqrt{n\log n}$ guaranteed by Shearer's result. For $G$ to not have larger independent sets requires both the average degree and maximum degree to be $(1/\sqrt{2}-o(1))\sqrt{n\log n}$. In such a graph we might expect there to be many disjoint independent sets of this size (indeed the main result of [DJPR18] says that this is also the average size of an independent set) and so the graph could be coloured more efficiently than by a naïve greedy procedure. This suggests a possible approach. Repeatedly remove large neighbourhoods and if the maximum degree is small, then we hope to colour the remaining graph with few colours. Johansson [Joh96], using a Rödl nibble, was the first to show a colouring version of Shearer's result: any triangle-free graph with maximum degree $\Delta$ has chromatic number $\mathcal{O}(\Delta/\log\Delta)$. The leading constant was improved by Pettie and Su [PS15] and finally Molloy [Mol19] matched Shearer's constant.

**Theorem 7.4** (Molloy)**.** *Any triangle-free graph of maximum degree $\Delta$ has* (*list*) *chromatic number at most $(1+o(1)) \cdot \Delta/\log\Delta$.*

We are now in a position to sketch the proof of Theorem 7.1. We will ignore all $o(1)$ terms. Let $G$ be an $n$-vertex triangle-free graph – we are trying to prove that $\chi(G) \leqslant 2\sqrt{n/\log n}$ and will assume the result holds for all smaller $n$. Firstly, if $G$ has maximum degree at most $d(n) := \sqrt{n\log n}$, then Molloy's theorem immediately gives the result. Otherwise, some vertex $v$ of $G$ has degree greater than $d(n)$. Let $G'$ be $G$ with all neighbours of $v$ deleted. As $G$ is triangle-free, the neighbourhood of $v$ is an independent set, so

$\chi(G) \leqslant \chi(G') + 1$. Also $n' = |G'| < n - d(n)$ and induction gives

$$\chi(G') \leqslant 2\sqrt{n'/\log n'}.$$

For this sketch, consider $\log n'$ and $\log n$ as identical so

$$\chi(G') \leqslant 2\sqrt{\frac{n-d(n)}{\log n}} \leqslant 2\sqrt{\frac{n}{\log n}} - 1,$$

where the final inequality follows by squaring both sides and cancelling terms. As $\chi(G) \leqslant \chi(G') + 1$, we are done. All that is needed for a full proof is to overcome the technical challenge of the $o(1)$ terms. We do this in § 7.4.

Consider trying the same proof strategy for the list chromatic number, in pursuit of Theorem 7.2. If there is a vertex of large degree, then we cannot simply colour its neighbourhood with one colour as there may be no colour appearing on the lists of all its neighbours. In place of degree we use the notion of *colour-degree*.

**Definition 7.5.** *Let $G$ be a graph with list assignment $L$. For a vertex $v$ and a colour $c \in L(v)$, the* ***colour-degree of*** *$c$* ***at*** *$v$ is*

$$\deg_L(v, c) = |\{u \colon uv \in E(G),\ c \in L(u)\}|,$$

*and the* ***maximum colour-degree at*** *$v$ is*

$$\Delta_L(v) = \max_{c \in L(v)} \deg_L(v, c).$$

If some colour-degree, say $\deg_L(v, c)$, is large, then we colour the neighbours of $v$ whose lists contain $c$ with colour $c$, remove $c$ from all other lists and delete the coloured vertices. What remains is a graph $G'$ of order $n - \deg_L(v, c)$ with $\chi_\ell(G) \leqslant \chi_\ell(G') + 1$. We bound $\chi_\ell(G')$ by induction. On the other hand, if all colour-degrees are small, we would like a colour-degree version of Molloy's theorem. The heuristic for why such a result might be true is that giving a colour to a vertex $v$ affects the lists of at most $\Delta_L(v)$ other vertices and so it is this, rather than the maximum degree, that ought to be the natural parameter for list colouring results.

Amini and Reed [AR08] were the first to prove such a result. For a list assignment $L$ on a triangle-free graph $G$, let $\Delta_L = \max_v \Delta_L(v)$. They showed that if the lists $L(v)$ all have size $\Omega(\Delta_L / \log \Delta_L)$, then $G$ is $L$-colourable (and claimed a leading constant of 1000 sufficed). More recently, Alon and Assadi [AA20] improved the leading constant to eight.

**Theorem 7.6** (Alon-Assadi)**.** *The following holds for all sufficiently large $d$. Let $G$ be a triangle-free graph with list assignment $L$. If, for every vertex $v$,*

$$
\begin{aligned}
|L(v)| &\geqslant 8d/\log d, \\
\Delta_L(v) &\leqslant d,
\end{aligned}
$$

*then $G$ is $L$-colourable.*

If the factor of eight could be replaced with $1+o(1)$, then the bound for the list chromatic number would match those for the fractional and usual chromatic number, that is, be $\sqrt{2}$ away from the bound for the Hall ratio.

We now discuss the edge results given in Theorem 7.3. Our upper bound for $f(n)$ can be converted to an improved upper bound for $g(m)$ via a tactic of Gimbel and Thomassen [GT00]. We partition the vertices by their degree. Those with small degree can be coloured using Molloy's theorem, while there cannot be too many with large degree (as there are only $m$ edges) and so the chromatic number of these can be coloured using our bound for $f(n)$. Here we have used the additivity of the chromatic number: $\chi(H_1+H_2) = \chi(H_1)+\chi(H_2)$. Such additivity certainly does not hold for the list chromatic number (consider the join of two independent sets). However, it almost does when the number of colours on each list is sufficiently large compared to the number of vertices: randomly partition the colours into two parts and promise to only use the colours from the first part on $H_1$ and the colours from the second part on $H_2$ (this effectively partitions each list). Provided the lists have size $\Omega(\log n)$, a Chernoff bound shows that no part of a list will be much less than half the original list size and so $\chi_\ell(H_1+H_2) \leqslant (1+o(1))(\chi_\ell(H_1)+\chi_\ell(H_2))$.

## 7.4 The Proofs

In this section we give the proofs of Theorems 7.1 to 7.3. We stress that the main ideas appeared in § 7.3, and what remains is a technical exercise.

*Proof of Theorem 7.1.* Let $f(x) = (2+A(x))\sqrt{x/\log x}$ where $A = o(1)$ is smooth and non-increasing (specified more precisely later). It suffices to show that any $n$-vertex triangle-free graph $G$ has chromatic number at most $f(n)$. We will induct upon $n$, and we may choose $A$ so that the theorem holds for all $n \leqslant 20$. Assume from now on that $n \geqslant 20$.

First suppose that every vertex of $G$ has degree at most $d(n) = \sqrt{n\log n}$. Theorem 7.4

gives an $\varepsilon(x) = o(1)$ such that

$$\begin{aligned}\chi(G) &\leqslant (1+\varepsilon(n))\frac{d(n)}{\log d(n)} \leqslant (1+\varepsilon(n))\frac{d(n)}{\log(n^{1/2})}\\ &= 2(1+\varepsilon(n))\sqrt{\frac{n}{\log n}}.\end{aligned}$$

Thus, we are done in this case provided

$$A(x) \geqslant 2\varepsilon(x). \tag{7.2}$$

In the second case there is some vertex $v$ with degree greater than $d(n)$. Let $G'$ be the graph obtained from $G$ by deleting all the neighbours of $v$. Then $G'$ has fewer than $n - d(n)$ vertices and

$$\chi(G) \leqslant \chi(G') + 1 \leqslant f(n - d(n)) + 1,$$

where the second inequality follows by induction. Hence, to complete the proof we need

$$f(n) - f(n - d(n)) \geqslant 1, \tag{7.3}$$

for all $n \geqslant 20$. It remains to check that it is possible to choose an $A$ such that (7.2) and (7.3) hold. We will assume that $A$ decays sufficiently slowly so that (7.2) holds.

The function $\sqrt{x/\log x}$ is concave for $x \geqslant 6$. If we choose $A$ decaying sufficiently slowly, then $f$ ought to be concave too. Indeed, if we choose $A$ so that $|A''(x)| \leqslant 1/(10x^2)$, then a quick calculation shows that for all $x \geqslant 10$, $f''(x) < 0$ (note that $A' \leqslant 0$). By concavity, for all $n \geqslant 20$,

$$f(n) - f(n - d(n)) \geqslant f'(n)d(n) = \left(1 + \tfrac{A(n)}{2}\right)\left(1 - \tfrac{1}{\log n}\right) + nA'(n).$$

Choosing $A$ so that $|A'(x)| \leqslant 1/(x\log x)$ and $A(x) \geqslant 8/\log x$ gives (7.3). One should worry that the conditions placed on the derivatives of $A$ might preclude it from tending to zero. Happily, integrating these shows that this is not the case. □

*Proof of Theorem 7.2.* Let $g(x) = (4\sqrt{2} + B(x))\sqrt{x/\log x}$ where $B = o(1)$ is smooth and non-increasing. We may choose $B$ so that Theorem 7.2 holds for all small $n$. Assume from now on that $n \geqslant n_0$ for some fixed $n_0$.

Let $L$ be a list assignment with $|L(v)| \geqslant g(n)$ for every vertex $v$. First suppose all colour-degrees are at most $d(n) = \sqrt{2}/4 \cdot \sqrt{n \log n}$. Now

$$\frac{8d(n)}{\log d(n)} \leqslant \frac{8d(n)}{\log(n^{1/2})} \leqslant g(n),$$

where the first inequality holds provided $n_0 \geqslant e^8$. Provided $n_0$ is large enough, Theorem 7.6 guarantees that $G$ is $L$-colourable.

Otherwise there is some vertex $v$ and colour $c \in L(v)$ with $\deg_L(v, c) > d(n)$. Let $G'$ be the graph obtained from $G$ by deleting all the neighbours of $v$ with colour $c$ on their list. Then $G'$ has fewer than $n - d(n)$ vertices and

$$\chi_\ell(G) \leqslant \chi_\ell(G') + 1 \leqslant g(n - d(n)) + 1.$$

We finish as in the proof of Theorem 7.1: choosing $B$ so that $|B''(x)| \leqslant 1/(10x^2)$ guarantees that $g$ is concave for $x \geqslant 10$. Thus for $n \geqslant 20$,

$$\begin{aligned} g(n) - g(n - d(n)) &\geqslant g'(n)d(n) \\ &= \left(1 + B(n) \cdot \tfrac{\sqrt{2}}{8}\right)\left(1 - \tfrac{1}{\log n}\right) + nB'(n) \cdot \tfrac{\sqrt{2}}{4}. \end{aligned}$$

Choosing $B$ so that $|B'(x)| \leqslant 1/(x \log x)$ and $B(x) \geqslant 32/\log x$ gives $\chi_\ell(G) \leqslant g(n)$, as required. $\square$

For the proof of Theorem 7.3, we need a standard estimate for the tail of binomial random variables (see, for example, Bollobás [Bol01, § 2.3]).

**Theorem 7.7.** *Let $X \sim \mathrm{Bin}(N, 1/2)$ be binomially distributed. Then, for any $\varepsilon \geqslant 0$,*

$$\mathbb{P}(X \leqslant (1 - \varepsilon)N/2) \leqslant e^{-\varepsilon^2 N/4}.$$

*Proof of Theorem 7.3.* Let $G$ be a triangle-free graph with $m$ edges. Set $d = (m \log m/3)^{1/3}$ and partition the vertices of $G$ as follows

$$\begin{aligned} U_1 &= \{v \in V(G) \colon \deg(v) \leqslant d\}, \\ U_2 &= V(G) \setminus U_1. \end{aligned}$$

First note that $G[U_1]$ has maximum degree at most $d$, so, by Molloy's theorem,

$$\chi(G[U_1]) \leqslant (1 + o(1))\frac{d}{\log d} = (3^{2/3} + o(1))\frac{m^{1/3}}{(\log m)^{2/3}}.$$

Next,

$$2m = \sum_{v \in G} \deg(v) \geqslant \sum_{v \in U_2} \deg(v) \geqslant d|U_2|,$$

so $|U_2| \leqslant 2 \cdot 3^{1/3} \cdot m^{2/3}/(\log m)^{1/3}$. Hence, by Theorem 7.1,

$$\chi(G[U_2]) \leqslant (2+o(1))\sqrt{\frac{|U_2|}{\log|U_2|}} \leqslant (2 \cdot 3^{2/3} + o(1))\frac{m^{1/3}}{(\log m)^{2/3}}.$$

Using $\chi(G) \leqslant \chi(G[U_1]) + \chi(G[U_2])$ gives the bound for $g(m)$.

We now prove the list chromatic bound. Let $G$ be a triangle-free graph with $m$ edges – we will assume throughout that $m$ is sufficiently large. Let $\varepsilon > 0$ and consider a list assignment $L$ giving each vertex a list of size at least

$$k = (1+\varepsilon)^2 \cdot 8 \cdot 6^{2/3}\frac{m^{1/3}}{(\log m)^{2/3}}.$$

It suffices to show that $G$ is $L$-colourable. We may assume that $G$ has no vertices of degree nought or one, as any such vertex can be removed before colouring $G$ and then compatible colours found when they are readded. In particular, we may assume that $n \leqslant m$.

For $D = (m \log m/6)^{1/3}$, consider the partition of $V(G)$ given by

$$\begin{aligned} V_1 &= \{v \in V(G) \colon \Delta_L(v) \leqslant D\}, \\ V_2 &= V(G) \setminus V_1. \end{aligned}$$

Let $\mathcal{C} = \cup_v L(v)$ be the set of all colours appearing on lists. For a partition $L_1 \cup L_2$ of $\mathcal{C}$, we will promise to colour the vertices of $V_1$ using only colours from $L_1$ and colour the vertices of $V_2$ using only colours from $L_2$. For each vertex $v$, let $L_i(v) = L(v) \cap L_i$. Suppose that, for every vertex $v$,

$$\min\{|L_1(v)|, |L_2(v)|\} \geqslant (4 \cdot 6^{2/3} + \varepsilon)\frac{m^{1/3}}{(\log m)^{2/3}}. \tag{7.4}$$

For every vertex $v \in V_1$: $\Delta_{L_1}(v) \leqslant \Delta_L(v) \leqslant D$ and

$$\frac{8D}{\log D} \leqslant (4 \cdot 6^{2/3} + o(1))\frac{m^{1/3}}{(\log m)^{2/3}} \leqslant |L_1(v)|$$

for sufficiently large $m$. Hence, by Theorem 7.6, $G[V_1]$ is $L_1$-colourable.

Every vertex in $V_2$ has some colour-degree at least $D$ and so has degree at least $D$.

Hence,

$$2m = \sum_{v \in V(G)} \deg(v) \geqslant \sum_{v \in V_2} \deg(v) \geqslant D|V_2|,$$

giving $|V_2| \leqslant 2 \cdot 6^{1/3} \cdot m^{2/3}/(\log m)^{1/3}$. Hence, by Theorem 7.2,

$$\chi_\ell(G[V_2]) \leqslant (4\sqrt{2} + o(1))\sqrt{\frac{|V_2|}{\log|V_2|}} \leqslant (4 \cdot 6^{2/3} + o(1))\frac{m^{1/3}}{(\log m)^{2/3}} \leqslant |L_2(v)|,$$

so $G[V_2]$ is $L_2$-colourable. As $L_1$ and $L_2$ are disjoint sets of colours, these can be combined to give an $L$-colouring of $G$.

We finally check that there is a partition $\mathcal{C} = L_1 \cup L_2$ for which (7.4) holds. For each colour $c \in \mathcal{C}$ we assign $c$ to $L_1$ with probability $1/2$ and to $L_2$ otherwise. We do this independently for each colour. By Theorem 7.7, for each vertex $v$,

$$\begin{aligned} \mathbb{P}\left(|L_i(v)| \leqslant (4 \cdot 6^{2/3} + \varepsilon)\frac{m^{1/3}}{(\log m)^{2/3}}\right) &\leqslant \mathbb{P}(|L_i(v)| \leqslant (1-\varepsilon)k/2), \\ &\leqslant e^{-\varepsilon^2 k/4}. \end{aligned}$$

As $n \leqslant m$, we have $k > n^{1/3}/(\log n)^{2/3}$ and so $e^{-\varepsilon^2 k/4} = o(1/n)$. A union bound now gives the existence of a partition $\mathcal{C} = L_1 \cup L_2$ satisfying (7.4). □

## 7.5 Concluding Remarks

Let $G$ be an $n$-vertex triangle-free graph and let its minimum and maximum degree be $\delta$ and $\Delta$ respectively. Theorem 7.1 gives the same upper bound for both $\chi(G)$ and $\chi_f(G)$ and our proof of Theorem 7.2, if supplied with a version of Alon-Assadi's theorem that had leading coefficient $(1 + o(1))$, would also match this. These are all a factor of $\sqrt{2}$ away from the upper bound for the Hall ratio given by Shearer [She83]. That was obtained by comparing two good lower bounds for the independence number, namely $\Delta$ and $(1 - o(1)) \cdot n \log \Delta/\Delta$.

Consider trying to remove the factor of $\sqrt{2}$ – we initially focus on the fractional chromatic number as it is smallest. The corresponding bounds to those used for the Hall ratio are Molloy's theorem ($\chi_f \leqslant (1 + o(1))\Delta/\log\Delta$) and taking a uniformly random neighbourhood (which shows that $\chi_f \leqslant n/\delta$). If $G$ is regular, so $\delta = \Delta$, then comparing these does remove the $\sqrt{2}$ factor giving the upper bound $\chi_f \leqslant (1 + o(1))\sqrt{2n/\log n}$. A uniformly random neighbourhood $\boldsymbol{I}$ satisfies $\mathbb{P}(v \in \boldsymbol{I}) = \deg(v)/n$ so gives greater weight to larger degree vertices. Thus, for $G$ non-regular, one needs a version of Molloy's theorem that gives greater weight to lower degree vertices. Such a 'local' result was conjectured by Kelly and Postle [KP18]: they conjectured there is a distribution on the independent sets of $G$ giving a random independent set $\boldsymbol{I}'$ satisfying

$$\mathbb{P}\big(v \in \boldsymbol{I}'\big) \geqslant (1 - o(1)) \frac{\log \deg(v)}{\deg(v)}.$$

Mixing $\boldsymbol{I}$ and $\boldsymbol{I}'$ (sampling a random independent set that is $\boldsymbol{I}$ with probability $1/2$ and $\boldsymbol{I}'$ otherwise) does then attain the bound $\chi_f \leqslant (1 + o(1))\sqrt{2n/\log n}$.

It is less clear how to bridge the gap for the (list) chromatic number. While there are some local versions of Molloy's theorem (for example, Davies, de Joannis de Verclos, Kang, and Pirot [DdKP20] give a local list colouring result where the list at a vertex $v$ only need have size $(1 + o(1)) \deg(v)/\log \deg(v)$), there is no version of taking a random neighbourhood nor a way to 'mix' any such bounds.

# Bibliography


[AKS80] Miklós Ajtai, János Komlós, and Endre Szemerédi; *A note on Ramsey numbers*; Journal of Combinatorial Theory, Series A **29**(3), November 1980: pages 354–360; DOI: 10.1016/0097-3165(80)90030-8. ↑29, 146, 147, 149

[All10] Peter Allen; *Dense H-free graphs are almost* $(\chi(H) - 1)$*-partite*; Electronic Journal of Combinatorics **17**(1.21), January 2010; DOI: 10.37236/293. ↑111

[ABG+13] Peter Allen, Julia Böttcher, Simon Griffiths, Yoshiharu Kohayakawa, and Robert Morris; *The chromatic thresholds of graphs*; Advances in Mathematics **235**, March 2013: pages 261–295; DOI: 10.1016/j.aim.2012.11.016. ↑27, 28, 29, 30, 90, 92, 127

[ABG+17] Peter Allen, Julia Böttcher, Simon Griffiths, Yoshiharu Kohayakawa, and Robert Morris; *Chromatic thresholds in dense random graphs*; Random Structures & Algorithms **51**(2), May 2017: pages 185–214; DOI: 10.1002/rsa.20708. ↑126, 130

[ABKR18] Peter Allen, Julia Böttcher, Yoshiharu Kohayakawa, and Barnaby Roberts; *Triangle-free subgraphs of random graphs*; Combinatorics, Probability and Computing **27**(2), March 2018: pages 141–161; DOI: 10.1017/S0963548317000219. ↑131

[Alo92] Noga Alon; *Choice numbers of graphs: a probabilistic approach*; Combinatorics, Probability and Computing **1**(2), 1992: pages 107–114; DOI: 10.1017/S0963548300000122. ↑149

[Alo96] Noga Alon; *Independence numbers of locally sparse graphs and a Ramsey type problem*; Random Structures & Algorithms **9**(3), October 1996: pages 271–278; DOI: 10.1002/(SICI)1098-2418(199610)9:3<271::AID-RSA1>3.0.CO;2-U. ↑29

[AA20] Noga Alon and Sepehr Assadi; *Palette sparsification beyond* $(\Delta + 1)$ *vertex coloring*; June 2020; arXiv: 2006.10456. ↑151

[AS06] Noga Alon and Benny Sudakov; *H-free graphs of large minimum degree*; Electronic Journal of Combinatorics **13**(1.19), March 2006; DOI: 10.37236/1045. ↑111

[AR08] Omid Amini and Bruce Reed; *List colouring constants of triangle free graphs*; Electronic Notes in Discrete Mathematics **30**, February 2008: pages 135–140; DOI: 10.1016/j.endm.2008.01.024. ↑151

[And62] Béla Andrásfai; *Über ein Extremalproblem der Graphentheorie*; Acta Mathematica Academiae Scientiarum Hungarica **13**, September 1962: pages 443–455; DOI: 10.1007/BF02020809. ↑28, 35

[AES74] Béla Andrásfai, Paul Erdős, and Vera T. Sós; *On the connection between chromatic number, maximal clique and minimal degree of a graph*; Discrete Mathematics **8**(3), 1974: pages 205–218; DOI: 10.1016/0012-365X(74)90133-2. ↑4, 118

[Bár78] Imre Bárány; *A short proof of Kneser's conjecture*; Journal of Combinatorial Theory, Series A **25**(3), November 1978: pages 325–326; DOI: 10.1016/0097-3165(78)90023-7. ↑92

[Beh46] Felix Adalbert Behrend; *On sets of integers which contain no three terms in arithmetical progression*; Proceedings of the National Academy of Sciences **32**(12), 1946: pages 331–332; DOI: 10.1073/pnas.32.12.331. ↑128

[BK21] Tom Bohman and Peter Keevash; *Dynamic concentration of the triangle-free process*; Random Structures & Algorithms **58**(2), 2021: pages 221–293; DOI: 10.1002/rsa.20973. ↑10, 146, 148, 150

[Bol01] Béla Bollobás; *Random graphs*; 2nd edition; Cambridge Studies in Advanced Mathematics; Cambridge University Press, 2001; DOI: 10.1017/CBO9780511814068. ↑21, 22, 134, 154

[BKT10] Julia Böttcher, Yoshiharu Kohayakawa, and Anusch Taraz; *Almost spanning subgraphs of random graphs after adversarial edge removal*; March 2010; arXiv: 1003.0890. ↑133

[Bra02] Stephan Brandt; *A 4-colour problem for dense triangle-free graphs*; Discrete Mathematics **251**(1), May 2002: pages 33–46; DOI: 10.1016/S0012-365X(01)00340-5. ↑35

[Bra03] Stephan Brandt; *Note: on the structure of graphs with bounded clique number*; Combinatorica **23**, December 2003: pages 693–696; DOI: 10.1007/s00493-003-0042-z. ↑4

[BP98] Stephan Brandt and Tomaž Pisanski; *Another infinite sequence of dense triangle-free graphs*; Electronic Journal of Combinatorics **5**(43), July 1998; DOI: 10.37236/1381. ↑35

[BT05] Stephan Brandt and Stéphan Thomassé; *Dense triangle-free graphs are four-colorable: a solution to the Erdős-Simonovits problem*; 2005; URL: http://perso.ens-lyon.fr/stephan.thomasse/; Submitted. ↑28, 34, 35

[CdKP20] Wouter Cames van Batenburg, Rémi de Joannis de Verclos, Ross J. Kang, and François Pirot; *Bipartite induced density in triangle-free graphs*; The Electronic Journal of Combinatorics **27**(2.34), May 2020; DOI: 10.37236/8650. ↑10, 11, 147, 148

[CJK97] C. C. Chen, Guoping P. Jin, and K. M. Koh; *Triangle-free graphs with large degree*; Combinatorics, Probability and Computing **6**(4), December 1997: pages 381–396; DOI: 10.1017/S0963548397003167. ↑35

[Con09] David Conlon; *A new upper bound for diagonal Ramsey numbers*; Annals of Mathematics **170**(2), September 2009: pages 941–960; DOI: 10.4007/annals.2009.170.941. ↑9

[Con14] David Conlon; *Combinatorial theorems relative to a random set*; April 2014; arXiv: 1404.3324. ↑129

[CGSS14] David Conlon, William Timothy Gowers, Wojciech Samotij, and Mathias Schacht; *On the KŁR conjecture in random graphs*; Israel Journal of Mathematics **203**(1), 2014: pages 535–580; DOI: 10.1007/s11856-014-1120-1. ↑133

[DdKP20] Ewan Davies, Rémi de Joannis de Verclos, Ross J. Kang, and François Pirot; *Coloring triangle-free graphs with local list sizes*; Random Structures & Algorithms **57**(3), 2020: pages 730–744; DOI: 10.1002/rsa.20945. ↑157

[DI21] Ewan Davies and Freddie Illingworth; *The $\chi$-Ramsey problem for triangle-free graphs*; July 2021; arXiv: 2107.12288; Submitted. ↑145

[DJPR18] Ewan Davies, Matthew Jenssen, Will Perkins, and Barnaby Roberts; *On the average size of independent sets in triangle-free graphs*; Proceedings of the American Mathematical Society **146**(1), January 2018: pages 111–124; DOI: 10.1090/proc/13728. ↑150

[Erd47] Paul Erdős; *Some remarks on the theory of graphs*; Bulletin of the American Mathematical Society **53**(4), 1947: pages 292–294; DOI: 10.1090/S0002-9904-1947-08785-1. ↑9

[Erd59] Paul Erdős; *Graph theory and probability*; Canadian Journal of Mathematics **11**, 1959: pages 34–38; DOI: 10.4153/CJM-1959-003-9. ↑3

[Erd64] Paul Erdős; *On extremal problems of graphs and generalized graphs*; Israel Journal of Mathematics **2**(3), September 1964: pages 183–190; DOI: 10.1007/BF02759942. ↑126

[Erd67a] Paul Erdős; *Some recent results on extremal problems in graph theory*; in: *Theory of Graphs* (International Symposium, Rome, 1966); Gordon and Breach, 1967: pages 117–123. ↑2, 14

[Erd67b] Paul Erdős; *Some remarks on chromatic graphs*; Colloquium Mathematicum **16**, 1967: pages 253–256; DOI: 10.4064/cm-16-1-253-256. ↑9, 145

[Erd68] Paul Erdős; *On some new inequalities concerning extremal properties of graphs*; in: *Theory of Graphs* (Colloquium, Tihany, 1966); Academic Press, 1968: pages 77–81. ↑2, 14

[EH85] Paul Erdős and András Hajnal; *Chromatic number of finite and infinite graphs and hypergraphs*; Discrete Mathematics **53**, March 1985: pages 281–285; DOI: 10.1016/0012-365X(85)90148-7. ↑10, 146, 147, 150

[ES66] Paul Erdős and Miklós Simonovits; *A limit theorem in graph theory*; Studia Scientiarum Mathematicarum Hungarica **1**, 1966: pages 51–57. ↑2

[ES73] Paul Erdős and Miklós Simonovits; *On a valence problem in extremal graph theory*; Discrete Mathematics **5**(4), 1973: pages 323–334; DOI: 10.1016/0012-365X(73)90126-X. ↑4, 27, 113

[ES46] Paul Erdős and Arthur Harold Stone; *On the structure of linear graphs*; Bulletin of the American Mathematical Society **52**, 1946: pages 1087–1091; DOI: 10.1090/S0002-9904-1946-08715-7. ↑2, 18

[ES35] Paul Erdős and George Szekeres; *A combinatorial problem in geometry*; Compositio Mathematica **2**, 1935: pages 463–470; DOI: 10.1007/978-0-8176-4842-8_3. ↑9, 21

[ET90] Paul Erdős and Prasad Tetali; *Representations of integers as the sum of k terms*; Random Structures & Algorithms **1**(3), 1990: pages 245–261; DOI: 10.1002/rsa.3240010302. ↑135

[EGM19] Beka Ergemlidze, Ervin Győri, and Abhishek Methuku; *Turán number of an induced complete bipartite graph plus an odd cycle*; Combinatorics, Probability and Computing **28**(2), March 2019: pages 241–252; DOI: 10.1017/S0963548318000354. ↑3, 17

[FGM20] Gonzalo Fiz Pontiveros, Simon Griffiths, and Robert Morris; *The triangle-free process and the Ramsey number $R(3,k)$*; Memoirs of the American Mathematical Society **263**(1274), 2020; DOI: 10.1090/memo/1274. ↑10, 146, 148, 150

[FR86] Péter Frankl and Vojtěch Rödl; *Large triangle-free subgraphs in graphs without $K_4$*; Graphs and Combinatorics **2**, December 1986: pages 135–144; DOI: 10.1007/BF01788087. ↑7

[Für96] Zoltán Füredi; *New asymptotics for bipartite Turán numbers*; Journal of Combinatorial Theory, Series A **75**(1), 1996: pages 141–144; DOI: 10.1006/jcta.1996.0067. ↑15

[FS13] Zoltán Füredi and Miklós Simonovits; *The history of degenerate (bipartite) extremal graph problems*; in: László Lovász, Imre Z. Ruzsa, and Vera T. Sós, editors, *Erdős Centennial*; Springer, 2013: pages 169–264; DOI: 10.1007/978-3-642-39286-3_7. ↑2

[GT00] John Gimbel and Carsten Thomassen; *Coloring triangle-free graphs with fixed size*; Discrete Mathematics **219**(1), May 2000: pages 275–277; DOI: 10.1016/S0012-365X(00)00087-X. ↑148, 152

[GL10] Wayne Goddard and Jeremy Lyle; *Dense graphs with small clique number*; Journal of Graph Theory **66**(4), December 2010: pages 319–331; DOI: 10.1002/jgt.20505. ↑28, 89, 91

[Gre02] Joshua E. Greene; *A new short proof of Kneser's conjecture*; The American Mathematical Monthly **109**(10), December 2002: pages 918–920; DOI: 10.2307/3072460. ↑92

[GHS02] András Gyárfás, Alice Hubenko, and József Solymosi; *Large cliques in $C_4$-free graphs*; Combinatorica **22**(2), April 2002: pages 269–274; DOI: 10.1007/s004930200012. ↑3, 16

[Häg82] Roland Häggkvist; *Odd cycles of specified length in nonbipartite graphs*; Annals of Discrete Mathematics **13**, 1982: pages 89–99; DOI: 10.1016/S0304-0208(08)73552-7. ↑28, 34

[Hol20] Andreas Fukami Holmsen; *Large cliques in hypergraphs with forbidden substructures*; Combinatorica **40**(4), March 2020: pages 527–537; DOI: 10.1007/s00493-019-4169-y. ↑3, 16, 22

[Ill20] Freddie Illingworth; *The chromatic profile of locally bipartite graphs*; December 2020; arXiv: 2012.10409; Submitted. ↑27

[Ill21a] Freddie Illingworth; *A note on induced Turán numbers*; May 2021; arXiv: 2105.12503; Submitted. ↑14

[Ill21b] Freddie Illingworth; *Graphs with no induced $K_{2,t}$*; Electronic Journal of Combinatorics **28**(1.19), January 2021; DOI: 10.37236/9223. ↑14

[Ill21c] Freddie Illingworth; *Minimum degree stability of H-free graphs*; February 2021; arXiv: 2102.11104; Submitted. ↑110

[Ill21d] Freddie Illingworth; *The chromatic profile of locally colourable graphs*; February 2021; arXiv: 2102.05522; Submitted. ↑89

[JT94] Tommy R. Jensen and Bjarne Toft; *Graph coloring problems*; John Wiley & Sons, 1994; DOI: 10.1002/9781118032497. ↑146

[Jin93] Guoping P. Jin; *Triangle-free graphs with high minimal degrees*; Combinatorics, Probability and Computing **2**(4), December 1993: pages 479–490; DOI: 10.1017/S0963548300000845. ↑35

[Jin95] Guoping P. Jin; *Triangle-free four-chromatic graphs*; Discrete Mathematics **145**(1), October 1995: pages 151–170; DOI: 10.1016/0012-365X(94)00063-O. ↑28, 35

[JLX02] Guoping P. Jin, Zhenhong Liu, and Baoguang Xu; *Odd wheels in graphs*; Journal of Combinatorial Theory, Series B **84**(2), March 2002: pages 213–224; DOI: 10.1006/jctb.2001.2073. ↑29

[Joh96] Anders Johansson; *Asymptotic choice number for triangle-free graphs*; technical report 91-5; DIMACS, 1996. ↑150

[KS20a] Tomáš Kaiser and Matěj Stehlík; *Edge-critical subgraphs of Schrijver graphs*; Journal of Combinatorial Theory, Series B **144**, September 2020: pages 191–196; DOI: 10.1016/j.jctb.2020.02.004. ↑93

[KS20b] Tomáš Kaiser and Matěj Stehlík; *Edge-critical subgraphs of Schrijver graphs II: the general case*; July 2020; arXiv: 2007.09204. ↑93

[Kee11] Peter Keevash; *Hypergraph Turán problems*; in: Robin Chapman, editor, *Surveys in Combinatorics 2011*; London Mathematical Society Lecture Note Series; Cambridge University Press, 2011: pages 83–140; DOI: 10.1017/CBO9781139004114.004. ↑2

[KP18] Tom Kelly and Luke Postle; *Fractional coloring with local demands*, November 2018; arXiv: 1811.11806. ↑156

[Kim95] Jeong Han Kim; *The Ramsey number $R(3,t)$ has order of magnitude $t^2/\log t$*; Random Structures & Algorithms **7**(3), October 1995: pages 173–207; DOI: 10.1002/rsa.3240070302. ↑146, 147, 148

[Kne55] Martin Kneser; *Aufgabe 360*; Jahresbericht der Deutschen Mathematiker-Vereinigung **58**(2), 1955. ↑92

[Koh97] Yoshiharu Kohayakawa; *Szemerédi's regularity lemma for sparse graphs*; in: F. Cucker and M. Shub, editors, *Foundations of computational mathematics*; Springer, 1997: pages 216–230; DOI: 10.1007/978-3-642-60539-0_16. ↑132

[KŁR97] Yoshiharu Kohayakawa, Tomasz Łuczak, and Vojtěch Rödl; *On $K^4$-free subgraphs of random graphs*; Combinatorica **17**(2), June 1997: pages 173–213; DOI: 10.1007/BF01200906. ↑129

[KR03] Yoshiharu Kohayakawa and Vojtěch Rödl; *Szemerédi's regularity lemma and quasi-randomness*; in: Bruce A. Reed and Cláudia L. Sales, editors, *Recent advances in algorithms and combinatorics*; CMS Books Mathematics; Springer, May 2003: pages 289–351; DOI: 10.1007/0-387-22444-0_9. ↑133

[KS96] János Komlós and Miklós Simonovits; *Szemerédi's regularity lemma and its applications in graph theory*; in: *Combinatorics, Paul Erdős is Eighty, Volume 2*; Bolyai Society Mathematical Studies **2**; János Bolyai Mathematical Society, Budapest, 1996: pages 295–352. ↑117, 126

[KST54] Támas Kővári, Vera Sós, and Pál Turán; *On a problem of K. Zarankiewicz*; Colloquium Mathematicae **3**(1), 1954: pages 50–57; DOI: 10.4064/cm-3-1-50-57. ↑2

[LTTZ18] Po-Shen Loh, Michael Tait, Craig Timmons, and Rodrigo M. Zhou; *Induced Turán numbers*; Combinatorics, Probability and Computing **27**(2), March 2018: pages 274–288; DOI: 10.1017/S0963548317000542. ↑2, 3, 14, 16

[Lov78] László Lovász; *Kneser's conjecture, chromatic number, and homotopy*; Journal of Combinatorial Theory, Series A **25**(3), 1978: pages 319–324; DOI: 10.1016/0097-3165(78)90022-5. ↑92

[Łuc06] Tomasz Łuczak; *On the structure of triangle-free graphs of large minimum degree*; Combinatorica **26**(4), August 2006: pages 489–493; DOI: 10.1007/s00493-006-0028-8. ↑36

[ŁT10] Tomasz Łuczak and Stéphan Thomassé; *Coloring dense graphs via VC-dimension*; July 2010; arXiv: 1007.1670. ↑5, 29, 30, 90, 92

[Man07] Willem Mantel; *Problem 28: solution by H. Gouwentak, W. Mantel, J. Teixeira de Mattes, F. Schuh and W. A. Wythoff*; Wiskundige Opgaven **10**, 1907: pages 60–61. ↑1, 7

[Mol19] Michael Molloy; *The list chromatic number of graphs with small clique number*; Journal of Combinatorial Theory, Series B **134**, January 2019: pages 264–284; DOI: 10.1016/j.jctb.2018.06.007. ↑150

[Nik10] Vladimir Nikiforov; *Chromatic number and mimimum degree of $K_r$-free graphs*; January 2010; arXiv: 1001.2070. ↑28, 89, 91

[NTT18] Vladimir Nikiforov, Michael Tait, and Craig Timmons; *Degenerate Turán problems for hereditary properties*; Electronic Journal of Combinatorics **25**(4.39), November 2018; DOI: 10.37236/6775. ↑3, 17

[Nil00] Alon Nilli; *Triangle-free graphs with large chromatic numbers*; Discrete Mathematics **211**(1), January 2000: pages 261–262; DOI: 10.1016/S0012-365X(99)00109-0. ↑148

[PS15] Seth Pettie and Hsin-Hao Su; *Distributed coloring algorithms for triangle-free graphs*; Information and Computation **243**, August 2015: pages 263–280; DOI: 10.1016/j.ic.2014.12.018. ↑150

[PT94] Svatopluk Poljak and Zsolt Tuza; *Bipartite subgraphs of triangle-free graphs*; SIAM Journal on Discrete Mathematics **7**(2), May 1994: pages 307–313; DOI: 10.1137/S0895480191196824. ↑148

[Ram30] Frank Plumpton Ramsey; *On a problem of formal logic*; Proceedings of the London Mathematical Society **s2-30**(1), 1930: pages 264–286; DOI: 10.1112/plms/s2-30.1.264. ↑9, 18

[RS13] Vojtěch Rödl and Mathias Schacht; *Extremal results in random graphs*; in: László Lovász, Imre Z. Ruzsa, and Vera T. Sós, editors, *Erdős Centennial*; Springer, 2013: pages 535–583; DOI: 10.1007/978-3-642-39286-3_20. ↑129

[RS78] Imre Z. Ruzsa and Endre Szemerédi; *Triple systems with no six points carrying three triangles*; in: *Combinatorics, Volume II* (Fifth Hungarian Colloquium); Colloquia Mathematica Societatis János Bolyai **18**; North-Holland, 1978: pages 939–945. ↑128

[Sah20] Ashwin Sah; *Diagonal Ramsey via effective quasirandomness*; May 2020; arXiv: 2005.09251. ↑9

[ST15] David Saxton and Andrew Gordon Thomason; *Hypergraph containers*; Inventiones mathematicae **201**, September 2015: pages 925–992; DOI: 10.1007/s00222-014-0562-8. ↑149

[Sch78] Alexander Schrijver; *Vertex-critical subgraphs of Kneser graphs*; Nieuw Archief voor Wiskunde **26**(3), 1978: pages 454–461. ↑90, 92, 93

[She83] James B. Shearer; *A note on the independence number of triangle-free graphs*; Discrete Mathematics **46**(1), 1983: pages 83–87; DOI: 10.1016/0012-365X(83)90273-X. ↑10, 21, 29, 147, 149, 156

[She95] James B. Shearer; *On the independence number of sparse graphs*; Random Structures & Algorithms **7**(3), October 1995: pages 269–271; DOI: 10.1002/rsa.3240070305. ↑29

[Sim68] Miklós Simonovits; *A method for solving extremal problems in graph theory, stability problems*; in: *Theory of Graphs* (Colloquium, Tihany, 1966); Academic Press, New York, 1968: pages 279–319. ↑2, 14

[SS01] Miklós Simonovits and Vera T. Sós; *Ramsey-Turán theory*; Discrete Mathematics **229**(1), 2001: pages 293–340; DOI: 10.1016/S0012-365X(00)00214-4. ↑15

[Sós69] Vera T. Sós; *On extremal problems in graph theory*; in: *Combinatorial Structures and their Application* (Calgary International Conference); Gordon and Breach, June 1969: pages 407–410. ↑14

[Spe75] Joel Spencer; *Ramsey's theorem – a new lower bound*; Journal of Combinatorial Theory, Series A **18**(1), January 1975: pages 108–115; DOI: 10.1016/0097-3165(75)90071-0. ↑9

[Sze78] Endre Szemerédi; *Regular partitions of graphs*; in: *Problèmes combinatoires et théorie des graphes*; Colloquium International CNRS **260**; CNRS, 1978: pages 399–401. ↑111, 116, 132

[Tho88] Andrew Gordon Thomason; *An upper bound for some Ramsey numbers*; Journal of Graph Theory **12**(4), 1988: pages 509–517; DOI: 10.1002/jgt.3190120406. ↑9

[Tho02] Carsten Thomassen; *On the chromatic number of triangle-free graphs of large minimum degree*; Combinatorica **22**(4), October 2002: pages 591–596; DOI: 10.1007/s00493-002-0009-5. ↑28

[Tur41] Pál Turán; *Eine Extremalaufgabe aus der Graphentheorie*; Matematikai és Fizikai Lapok **48**, 1941: pages 436–452. ↑1, 5, 14